

An improved exceptional-set bound for the Erdős–Straus equation via saturated divisor fans

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Abstract

Let $E(N)$ be the number of integers $2 \leq n \leq N$ for which $4/n = 1/x + 1/y + 1/z$ has no solution in positive integers. Vaughan proved $E(N) \ll N \exp\{-c(\log N)^{2/3}\}$. We prove

$$E(N) \ll N \exp\{-c(\log N)^{3/4}\}.$$

The improvement comes from the Type II divisor fan: for each $a \geq 1$, every divisor of a^2 supplies a congruence class modulo $4a - 1$ of representable n , and the classical representing congruences are the boundary cases of this family. A conductor-separated subfan keeps a mass of order $(\log X)^3$, and a finite-interval sieve turns this mass into the stated saving at the scale $\log X \asymp (\log N)^{1/4}$.

We also record consequences for prime denominators, for propagation along large prime factors, and for the Erdős–Straus–Schinzel problem with fixed numerator $m \geq 2$, together with a geometric interpretation on the associated log $K3$ surfaces. Variants with growing numerator or at Elliott–Halberstam strength are stated under explicit additional hypotheses.

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1 Introduction and main theorem

The Erdős–Straus conjecture asserts that for every integer $n \geq 2$ the equation

$$\frac{4}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \tag{1}$$

has a solution in positive integers. Let $E(N)$ count the exceptions $2 \leq n \leq N$. Vaughan proved the classical estimate

$$E(N) \ll N \exp\{-c(\log N)^{2/3}\}$$

for some absolute $c > 0$ [16]. Our main result improves the exponent. The same construction also recovers a logarithmic factor that a one-rough-coordinate sieve loses, through the small-prime saturation device described in [Remark 12.1](#).

Theorem 1.1 (main theorem). *There is an absolute constant $c > 0$ such that, for all $N \geq 3$,*

$$E(N) \ll N \exp\{-c(\log N)^{3/4}\}.$$

The proof covers residue classes by congruence certificates. Each selected certificate gives a residue class modulo some q , and every integer in that class has an explicit representation. The difficulty is to keep cubic total weight while making the residual intersections sparse enough for Brun–Suen: small-prime factors are absorbed into the base class, while conductor separation keeps a cubic mass of residual events. We estimate the number of integers missed by all selected classes.

The unconditional consequences include the geometric form in [Corollary 2.5](#). In [Section 13](#) we also record an exceptional-prime bound with the same exponential saving, a large-prime-factor propagation theorem, and a fixed-numerator version for the Erdős–Straus–Schinzel equation.

The growing-numerator statements are separated from these consequences. They assume the uniform numerator-saturation package introduced in [Hypothesis 13.5](#), and the Elliott–Halberstam variant also assumes the reciprocal-prime distribution hypothesis in [Hypothesis 13.8](#). Under those stated assumptions, the growing-numerator theorem improves the Vaughan-type Pomerance–Weingartner exponent in a logarithmic range of numerators [13].

The structural starting point is the complete Type II divisor fan

$$4a - 1 \mid n + 4h, \quad h \mid a^2.$$

The three traditional boundary rays $h = 1, a, a^2$ recover the classical shifts appearing in the work of Mordell, Yamamoto, and Webb [12, 19, 18], but the interior divisors carry cubic logarithmic mass. We also prove the fixed-modulus no-obstruction statement for

this full cubic fan: for every fixed odd modulus M and every reduced class $c \pmod{M}$, the interior fan supplies cubic mass in the class compatible with c . Thus the local congruences singled out by the boundary tests are not local obstructions to the full fan.

Fixed slices are already classical in the sense of Elsholtz–Tao [5]. Here many interior divisors are used simultaneously, with conductor separation controlling the dependency graph.

The proof of [Theorem 1.1](#) uses a sparse subfamily of this fan. Write

$$e = rs^2, \quad \rho(e) = rs,$$

with r, s squarefree but not sieved at small primes. Split the medium factor as $d = d_- d_+$, where d_- is a small-prime part absorbed into a base residue class and d_+ is rough. We then choose moduli of the form

$$Q = d_- d_+ p,$$

where p is a large prime. After conditioning on a base class modulo $P(z)$ that absorbs the d_- congruence, the residual modulus is $q = d_+ p$. The parameter ranges force

$$\rho(e) > d.$$

This inequality gives the large-prime uniqueness input: two distinct compatible events cannot share their large prime p .

Vaughan’s exceptional-set sieve is the comparison point for the saving, while representation-count results situate the construction among related work on the Erdős–Straus equation. Elsholtz and Tao proved average-order bounds for the number of representations over primes [5], and Elsholtz–Planitzer and Bloom–Elsholtz give further results on representation counts and Egyptian fractions [6, 2]. Bright and Loughran studied the associated surfaces through the Brauer–Manin obstruction [3]. Pomerance and Weingartner recently treated exceptions to the Erdős–Straus–Schinzel generalization and generalized Vaughan-type almost-all results [13].

The proof here keeps Vaughan’s exceptional-set-covering viewpoint, but changes the certificate family. The dense divisor fan is thinned to a conductor-separated subfan, while the small-prime part of the medium coordinate is saturated. This keeps the logarithmic mass that a one-rough-coordinate sieve would lose.

The same result has a geometric interpretation. Let

$$U_n : \quad 4XYZ = n(XY + XZ + YZ) \subset \mathbb{A}^3$$

be the affine Erdős–Straus surface in the positive chamber, and let S_n be its projective cubic compactification

$$4XYZ = n(XYW + XZW + YZW) \subset \mathbb{P}^3.$$

Bright and Loughran identify these as affine cubic log $K3$ surfaces, all geometrically isomorphic after scaling, and study their Cayley nodal cubic compactification and desingularisation [3]. In this language, [Theorem 1.1](#) says that outside $O(N \exp\{-c(\log N)^{3/4}\})$ parameters $n \leq N$, the integral log surface U_n has an integral point in the positive real chamber. [Theorem 2.4](#) below shows that the divisor-fan certificates lie on explicit rational plane sections of the Cayley cubic.

[Section 6](#) collects the standard analytic estimates used below.

Formalization. A companion Lean 4 development [11] formalizes the deduction of [Theorem 1.1](#) from nine named theorems, either cited here or proved in the manuscript. It verifies the certificate identities, parameter inequalities, mass and tensorisation estimates, dependency bounds, finite transfer, optimization, and smooth lifting. The weighted reciprocal Bombieri–Vinogradov estimate and the long squarefree progression estimates are stated in the exact forms used here rather than derived from more primitive versions inside Lean; the repository records this boundary explicitly.

2 The Type II divisor fan

A Type II representation for a prime denominator is one in which exactly two denominators are divisible by the prime.

Theorem 2.1 (linear Type II certificate). *Let p be an odd prime. Suppose that positive integers d, e, A, B satisfy*

$$(4deA - 1)B = Ap + d. \tag{2}$$

Then

$$\frac{4}{p} = \frac{1}{pdeA} + \frac{1}{peAB} + \frac{1}{deB}.$$

Conversely, every Type II representation can be put in this form after permuting the two p -divisible denominators.

Proof. Putting the three displayed fractions over the common denominator $pdeAB$ gives numerator $B + d + Ap = 4deAB$ by (2). Conversely, write a Type II solution as

$$\frac{4}{p} = \frac{1}{pu} + \frac{1}{pv} + \frac{1}{w}, \quad p \nmid w.$$

Let $g = (u, v)$ and write $u = gd$, $v = gB$, where $(d, B) = 1$. Clearing denominators and dividing by g gives

$$4gdBw = w(d + B) + pgdB.$$

Reduction modulo d and modulo B shows that $dB \mid w$; write $w = deB$ with $e \geq 1$. Substitution and division by dB yield

$$4gdeB = e(d + B) + pg.$$

Thus $e \mid pg$. Since $p \nmid w$, we have $(e, p) = 1$, so $e \mid g$; write $g = eA$. Dividing the last identity by e now gives

$$4deAB = d + B + pA,$$

which is (2). The formulas $u = deA$, $v = eAB$, and $w = deB$ follow from the definitions. This is the normal form used in the Elsholtz–Tao Type II classification [5, Sec. 2]. $\square$

Put

$$a = deA, \quad h = d^2e, \quad q_a = 4a - 1.$$

Then $h \mid a^2$.

Theorem 2.2 (complete Type II divisor fan). *Let p be an odd prime and $a \geq 1$. There is a Type II solution with first denominator pa if and only if there exists a divisor $h \mid a^2$ such that*

$$4a - 1 \mid p + 4h. \quad (3)$$

When (3) holds, write $h = d^2e$ and $a = deA$. Then

$$B = \frac{Ap + d}{4deA - 1}$$

is integral, and Theorem 2.1 gives a Type II representation.

Proof. Starting from (2),

$$A(p + 4d^2e) = Ap + d + d(4deA - 1),$$

so $4deA - 1$ divides $A(p + 4d^2e)$. Since $(A, 4deA - 1) = 1$ because $4deA - 1 \equiv -1 \pmod{A}$, cancellation gives $4a - 1 \mid p + 4h$. Conversely, suppose $h \mid a^2$. Prime by prime, write $h = d^2e$ and $a = deA$ as follows. If $v_\ell(a) = \alpha$ and $v_\ell(h) = \beta \leq 2\alpha$, take

$$v_\ell(d) = \max(0, \beta - \alpha), \quad v_\ell(e) = \beta - 2v_\ell(d),$$

and put the remaining exponent into A . Then

$$Ap + d = A(p + 4h) - d(4a - 1).$$

Thus (3) implies $(4a - 1) \mid Ap + d$, and Theorem 2.1 applies. $\square$

For all integer denominators we need the following direct certificate.

Lemma 2.3 (divisor-fan certificate). *Let $n \geq 1$, let $a \geq 1$, let $q = 4a - 1$, and let e be a positive divisor of a^2 . If*

$$n \equiv -4e \pmod{q},$$

then $4/n$ is a sum of three positive unit fractions.

Proof. Put $A = an$. Since $4a \equiv 1 \pmod{q}$, the congruence on n gives $A \equiv -e \pmod{q}$. Also $(e, q) = 1$. Define

$$x = A, \quad y = \frac{A + e}{q}, \quad z = \frac{A + A^2/e}{q}.$$

The integer y is integral by $A \equiv -e \pmod{q}$. Since e is a unit modulo q and $A \equiv -e \pmod{q}$, one has $A^2/e \equiv e \pmod{q}$, so z is integral. Positivity is clear. Moreover

$$qy - A = e, \quad qz - A = A^2/e,$$

and hence $(qy - A)(qz - A) = A^2$. Expanding gives $qyz = A(y + z)$, so $1/y + 1/z = q/A$. Therefore

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1 + q}{an} = \frac{4}{n}.$$

$\square$

Theorem 2.4 (divisor-fan curves on the Cayley compactification). *Fix $n \geq 2$ and let $U_n : 4xyz = n(xy + xz + yz) \subset \mathbb{A}^3$. For every $a \geq 1$ put $q = 4a - 1$ and let*

$$C_{n,a} = U_n \cap \{x = an\}.$$

Then $C_{n,a}$ is the rational affine curve

$$qyz = an(y + z), \quad \text{equivalently} \quad (qy - an)(qz - an) = a^2n^2. \quad (4)$$

The map

$$u \mapsto \left(an, \frac{an + u}{q}, \frac{an + a^2n^2/u}{q} \right) \quad (5)$$

identifies $C_{n,a}$ birationally with $\mathbb{G}_m$, with inverse $u = qy - an$. The divisor-fan certificate of [Lemma 2.3](#) is exactly the integral subfamily obtained by taking $u = e$ with $e \mid a^2$ and $n \equiv -4e \pmod{q}$. Consequently the complete Type II fan lies on explicit rational plane sections in the Cayley cubic pencil; the parameter compactification of each such affine curve adds the two points corresponding to $u = 0$ and $u = \infty$.

Proof. Substituting $x = an$ into $4xyz = n(xy + xz + yz)$ gives

$$4ayz = an(y + z) + yz,$$

which is the first equation in (4). Multiplying it by $q = 4a - 1$ and completing the product gives

$$(qy - an)(qz - an) = q^2yz - anq(y + z) + a^2n^2 = a^2n^2.$$

Thus $u = qy - an$ is a rational parameter and $qz - an = a^2n^2/u$, proving (5). Conversely, every point of the displayed form satisfies (4) and hence lies on U_n . If $u = e \mid a^2$ and $n \equiv -4e \pmod{q}$, then $q \mid an + e$ because $4a \equiv 1 \pmod{q}$, and the second coordinate is integral. The same congruence gives integrality of the third coordinate, since $e \mid a^2$ and the calculation in [Lemma 2.3](#) shows $q \mid an + a^2n^2/e$. This is the divisor-fan point. The parameter u identifies the affine curve with $\mathbb{G}_m$ away from the usual exceptional parameter values, so its parameter compactification adds precisely the points $u = 0$ and $u = \infty$. $\square$

Corollary 2.5 (geometric form of the exceptional-set theorem). *There is an absolute constant $c > 0$ such that, for all $N \geq 3$, for all but*

$$O\left(N \exp\{-c(\log N)^{3/4}\}\right)$$

integers $n \leq N$, the log K3 surface U_n has an integral point in the positive chamber. Moreover, one may choose such a point on one of the rational fan curves $C_{n,a}$ from [Theorem 2.4](#).

Proof. This follows from [Theorem 1.1](#) and the certificate-to-curve identification in [Theorem 2.4](#). Bright–Loughran’s Brauer–Manin analysis concerns local and strong-approximation obstructions on each fixed fibre; the present corollary is complementary, bounding the density of fibres for which the explicit rational-curve sieve has not produced an integral point. $\square$

3 Fixed slices, boundary rays, and interior channels

The boundary divisors $h = 1, a, a^2$ recover the three classical one-shift constructions. Fixed interior slices reduce to known Type II congruence classes.

Proposition 3.1 (fixed-slice equivalence). *Fix positive integers d, e, A , put $a = deA$ and $f = 4deA - 1$. The one-parameter family obtained from*

$$fB = Ap + d$$

is an Elsholtz–Tao Type II congruence class. Equivalently, after eliminating B , it is the congruence $Ap \equiv -d \pmod{f}$, with denominators polynomial in p and $(Ap + d)/f$.

Proof. For fixed d, e, A , the three denominators in [Theorem 2.1](#) are polynomial in p and $B = (Ap + d)/f$. This is the fixed-parameter Type II setting of Elsholtz–Tao [[5](#), Sec. 2]. Reversing the descent in [Theorem 2.1](#) gives the converse. $\square$

Let $\mathcal{S}_1$ denote the set of integers in which every prime $\equiv 3 \pmod{4}$ occurs to even exponent. Define the boundary $\mathcal{S}_1$ -proxy set by

$$\mathcal{Q}_{\partial}(N) = \left\{ p \leq N : p \text{ prime, } 4p + 1 \in \mathcal{S}_1, \frac{p+1}{2} \in \mathcal{S}_1, p + 4 \in \mathcal{S}_1 \right\}.$$

This proxy records the three classical boundary tests, but it is not the literal set on which all three boundary rays fail: an integer in $\mathcal{S}_1$ may still contain a square of a prime $3 \pmod{4}$ and hence a usable boundary divisor. The elementary local envelope for the proxy is the following 24-class condition modulo 840.

Lemma 3.2 (boundary-proxy local envelope). *If $p > 7$ belongs to $\mathcal{Q}_\partial(N)$, then*

$$p \equiv 1 \pmod{24}.$$

Equivalently,

$$p \bmod 840 \in \mathcal{H} := \{r \in (\mathbb{Z}/840\mathbb{Z})^\times : r \equiv 1 \pmod{24}\}.$$

Proof. Since $p + 4 \in \mathcal{S}_1$, the integer $p + 4$ cannot be $3 \pmod{4}$; hence $p \equiv 1 \pmod{4}$. If $p \equiv 5 \pmod{8}$, then $(p + 1)/2 \equiv 3 \pmod{4}$, impossible for an element of $\mathcal{S}_1$. Therefore $p \equiv 1 \pmod{8}$.

It remains to prove $p \equiv 1 \pmod{3}$. Since $p > 3$, either $p \equiv 1$ or $2 \pmod{3}$. If $p \equiv 2 \pmod{3}$, then both $p + 1$ and $p + 4$ are divisible by 3, and their difference is 3. They cannot both be divisible by 9. Thus at least one of $(p + 1)/2$ and $p + 4$ has 3-adic valuation exactly 1, contradicting membership in $\mathcal{S}_1$. Hence $p \equiv 1 \pmod{3}$. Combining $p \equiv 1 \pmod{8}$ and $p \equiv 1 \pmod{3}$ gives $p \equiv 1 \pmod{24}$. For $p > 7$, primality also places p in $(\mathbb{Z}/840\mathbb{Z})^\times$. $\square$

Example 3.3 (interior channel inside the boundary proxy). *The prime $p = 2521$ belongs to the boundary $\mathcal{S}_1$ -proxy, and in this case the three boundary-ray divisibility tests fail. Indeed,*

$$4p + 1 = 5 \cdot 2017, \quad \frac{p+1}{2} = 13 \cdot 97, \quad p + 4 = 5^2 \cdot 101,$$

and every displayed prime factor is $1 \pmod{4}$. The interior divisor $a = 12$, $h = 16$ gives

$$4a - 1 = 47, \quad p + 4h = 2521 + 64 = 2585 = 47 \cdot 55.$$

Thus Theorem 2.2 gives a Type II solution from an interior divisor.

4 Cubic fan mass and the pairwise-coprime barrier

For $a \geq 1$ set

$$\Lambda_{\text{int}}(a) = \{h : h \mid a^2, h \notin \{1, a, a^2\}\}, \quad \Lambda_{<}(a) = \{h : h \mid a^2, 1 < h < a\}.$$

The full interior fan has cubic logarithmic mass.

Lemma 4.1 (Dirichlet series). *For $\Re s > 1$,*

$$\sum_{n=1}^{\infty} \frac{\tau(n^2)}{n^s} = \frac{\zeta(s)^3}{\zeta(2s)}.$$

Consequently,

$$\sum_{n \leq L} \frac{\tau(n^2)}{n} = \frac{1}{6\zeta(2)} (\log L)^3 + O((\log L)^2).$$

Proof. The Euler factor identity is

$$\sum_{k \geq 0} (2k+1)x^k = \frac{1+x}{(1-x)^2} = \frac{1-x^2}{(1-x)^3}.$$

Multiplying over primes gives the Dirichlet series. The asymptotic follows by Perron's formula, or by Selberg–Delange, from the triple pole of $\zeta(s)^3/\zeta(2s)$ at $s = 1$. $\square$

Proposition 4.2 (cubic divisor-fan mass). *As $L \rightarrow \infty$,*

$$\sum_{a \leq L} \frac{|\Lambda_{\text{int}}(a)|}{4a-1} \sim \frac{1}{24\zeta(2)} (\log L)^3.$$

Moreover,

$$\sum_{a \leq L} \frac{|\Lambda_{<}(a)|}{4a-1} \sim \frac{1}{48\zeta(2)} (\log L)^3.$$

Proof. The involution $h \mapsto a^2/h$ pairs divisors below and above a . The excluded divisors $1, a, a^2$ contribute only $O(\log L)$ to the weighted sum. Since

$$\frac{1}{4a-1} = \frac{1}{4a} + O(a^{-2}),$$

Lemma 4.1 gives the first asymptotic, and the lower-half asymptotic is half of it. $\square$

The preceding mass law is global. The corresponding fixed-modulus local law distinguishes analytic dependencies from fixed-modulus congruence obstructions. The interior fan carries cubic mass in every reduced class modulo a fixed modulus, so the boundary congruence tests of [Section 3](#) are not genuine local obstructions to representability.

Theorem 4.3 (full-fan local obstruction theorem). *Let M be a fixed odd modulus and let $c \in (\mathbb{Z}/M\mathbb{Z})^\times$. Put*

$$a_M \equiv 4^{-1} \pmod{M}, \quad u_c \equiv -c4^{-1} \pmod{M},$$

and define

$$\mathcal{M}_{M,c}(L) = \sum_{a \leq L : a \equiv a_M \pmod{M}} \frac{1}{4a-1} \#\{h \in \Lambda_{\text{int}}(a) : h \equiv u_c \pmod{M}\}.$$

Then, as $L \rightarrow \infty$,

$$\mathcal{M}_{M,c}(L) = \frac{\mathfrak{S}(M)}{24\varphi(M)^2} (\log L)^3 + O_M((\log L)^2), \quad (6)$$

where

$$\mathfrak{S}(M) = \frac{1}{\zeta(2)} \prod_{\ell|M} \frac{(1-\ell^{-1})^3}{1-\ell^{-2}} > 0.$$

The definition sums over divisors: distinct divisors that induce the same congruence modulo $4a-1$ are counted separately. For fixed M , the error term is uniform in the reduced class c . In particular, for each fixed odd prime ℓ , each fixed $k \geq 1$, and every $c \in (\mathbb{Z}/\ell^k\mathbb{Z})^\times$, the interior-divisor mass compatible with the denominator class $n \equiv c \pmod{\ell^k}$ has positive cubic order. At the prime 2 there is a separate Chinese remainder theorem (CRT) observation: because every modulus $4a-1$ is odd, an odd residue class $n \equiv c \pmod{2^k}$ can be imposed simultaneously with a congruence modulo $4a-1$. This observation concerns the denominator class, not the distribution of the moduli $4a-1$ among odd 2-adic classes.

Proof. Because a_M is a unit modulo M , every $a \equiv a_M \pmod{M}$ is coprime to M , and so is every divisor of a^2 . For a Dirichlet character $\chi \pmod{M}$, put

$$D_\chi(a) = \sum_{h|a^2} \chi(h).$$

If $q = 4a-1$, then $M \mid q$, while $h \equiv u_c \pmod{M}$ gives $-4h \equiv c \pmod{M}$. Moreover $(h, q) = 1$, since $h \mid a^2$ and $(a, 4a-1) = 1$. Thus every certificate class $n \equiv -4h \pmod{q}$ counted here is reduced and lies inside $n \equiv c \pmod{M}$.

Character orthogonality gives, for such a ,

$$\#\{h \mid a^2 : h \equiv u_c \pmod{M}\} = \frac{1}{\varphi(M)} \sum_{\chi \pmod{M}} \bar{\chi}(u_c) D_\chi(a),$$

while

$$\mathbf{1}_{a \equiv a_M \pmod{M}} = \frac{1}{\varphi(M)} \sum_{\psi \pmod{M}} \bar{\psi}(a_M) \psi(a).$$

Thus the required unweighted count is a finite linear combination, with coefficients of modulus at most $\varphi(M)^{-2}$, of the partial sums of $D_\chi(a)\psi(a)$.

For $\Re s > 1$, the corresponding Dirichlet series is

$$\sum_{(a,M)=1} \frac{D_\chi(a)\psi(a)}{a^s} = \frac{L^{(M)}(s, \psi) L^{(M)}(s, \chi\psi) L^{(M)}(s, \chi^2\psi)}{L^{(M)}(2s, \chi^2\psi^2)}. \quad (7)$$

Indeed, if $p \nmid M$, $x = p^{-s}$, and $D_\chi(p^\nu) = \sum_{j=0}^{2\nu} \chi(p)^j$, then

$$\sum_{\nu \geq 0} D_\chi(p^\nu) \psi(p)^\nu x^\nu = \frac{1 - \chi(p)^2 \psi(p)^2 x^2}{(1 - \psi(p)x)(1 - \chi(p)\psi(p)x)(1 - \chi(p)^2 \psi(p)x)}.$$

This is the local factor in (7); the notation $L^{(M)}$ records that the Euler factors at primes dividing M are absent.

The denominator in (7) is holomorphic and nonzero near $s = 1$. A numerator factor has a pole there precisely when its character is principal. If all three factors had poles, then ψ , $\chi\psi$, and $\chi^2\psi$ would all be principal. The first two conditions force $\psi = \chi = \chi_0$. Hence the principal pair is the unique pair with a triple pole; every other pair has pole order at most two. Equivalently, after extracting one factor of $\zeta(s)$ for each principal character among $\psi, \chi\psi, \chi^2\psi$, every series in (7) has the form

$$\zeta(s)^{r(\chi, \psi)} G_{\chi, \psi}(s), \quad 0 \leq r(\chi, \psi) \leq 3,$$

where $G_{\chi,\psi}$ is holomorphic in a fixed neighbourhood of $s = 1$. We verify the remaining hypotheses of the Selberg–Delange theorem. Put $f_{\chi,\psi}(a) = \mathbf{1}_{(a,M)=1} D_{\chi}(a)\psi(a)$, and let $F_{\chi,\psi}$ denote its Dirichlet series. Choose $c_0 > 0$ so small that

$$\Delta(c_0) = \left\{ \sigma + it : \sigma > 1 - \frac{c_0}{1 + \log(2 + |t|)} \right\} \subset \{ \sigma > 3/4 \}.$$

On this region, the principal factors left after extracting $\zeta(s)$ are finite Euler products, the nonprincipal Dirichlet L -functions are entire, and $L^{(M)}(2s, \chi^2 \psi^2)^{-1}$ has an absolutely convergent Euler product. Fixed-modulus convexity bounds therefore give

$$G_{\chi,\psi}(\sigma + it) \ll_M (1 + |t|)^{1/2} \quad (\sigma + it \in \Delta(c_0)).$$

The series converges absolutely for $\Re s > 1$, and its coefficients have the nonnegative majorant

$$|f_{\chi,\psi}(a)| \leq \mathbf{1}_{(a,M)=1} \tau(a^2).$$

Its Dirichlet series is

$$\sum_{(a,M)=1} \frac{\tau(a^2)}{a^s} = \frac{\zeta(s)^3}{\zeta(2s)} \prod_{\ell|M} \frac{(1 - \ell^{-s})^3}{1 - \ell^{-2s}},$$

whose quotient by $\zeta(s)^3$ is holomorphic and bounded on $\Delta(c_0)$. Thus $F_{\chi,\psi}$ belongs to the Selberg–Delange class

$$\mathcal{T}\left(r(\chi, \psi), 3, c_0, \frac{1}{2}, K_M\right)$$

for some K_M , uniformly over the finitely many character pairs. Dirichlet’s nonvanishing theorem gives $L(1, \eta) \neq 0$ for every nonprincipal character η , so $r(\chi, \psi)$ is the pole order. For the principal pair, the majorant series above is the series itself, and therefore

$$\lim_{s \rightarrow 1} (s-1)^3 \sum_{(a,M)=1} \frac{\tau(a^2)}{a^s} = \frac{1}{\zeta(2)} \prod_{\ell|M} \frac{(1 - \ell^{-1})^3}{1 - \ell^{-2}} = \mathfrak{S}(M).$$

The fixed-modulus Selberg–Delange theorem [15, Thm. II.5.2], applied to each of the finitely many character pairs, now gives

$$A_{M,c}(L) := \sum_{\substack{a \leq L \\ a \equiv a_M \pmod{M}}} \#\{h \mid a^2 : h \equiv u_c \pmod{M}\} = C_M L (\log L)^2 + O_M(L \log L), \quad (8)$$

uniformly in c , where

$$C_M = \frac{\mathfrak{S}(M)}{2\varphi(M)^2}.$$

More precisely, the Laurent term $\mathfrak{S}(M)(s-1)^{-3}$ contributes $\mathfrak{S}(M)L(\log L)^2/\Gamma(3)$, and $\Gamma(3) = 2$. All nonprincipal character pairs contribute $O_M(L \log L)$ because their pole order is at most two.

Stieltjes partial summation in (8) gives

$$\begin{aligned} \sum_{\substack{a \leq L \\ a \equiv a_M \pmod{M}}} \frac{\#\{h \mid a^2 : h \equiv u_c \pmod{M}\}}{a} &= \frac{A_{M,c}(L)}{L} + \int_1^L \frac{A_{M,c}(t)}{t^2} dt \\ &= \frac{C_M}{3} (\log L)^3 + O_M((\log L)^2) \\ &= \frac{\mathfrak{S}(M)}{6\varphi(M)^2} (\log L)^3 + O_M((\log L)^2). \end{aligned}$$

Moreover,

$$\left| \frac{1}{4a-1} - \frac{1}{4a} \right| \ll \frac{1}{a^2},$$

and $\sum_{a \geq 1} \tau(a^2)/a^2 < \infty$. Replacing $1/a$ by $4/(4a-1)$ therefore changes the last sum by $O(1)$. Finally, removing the at most three divisors $1, a, a^2$ changes the sum by

$$O\left(\sum_{\substack{a \leq L \\ a \equiv a_M \pmod{M}}} \frac{1}{4a-1}\right) = O_M(\log L).$$

After the factor $1/4$ is restored, this proves (6). Taking $M = \ell^k$ gives the assertion for odd ℓ , while the statement at 2 follows from the CRT observation in the theorem. $\square$

Pairwise-coprime moduli do not retain the full cubic mass. The sparse family in [Section 5](#) instead allows controlled medium overlaps while forbidding shared large primes.

Lemma 4.4 (weighted Titchmarsh divisor bound). *For $L \geq 3$,*

$$\sum_{\substack{a \leq L \\ 4a-1 \text{ prime}}} \frac{\tau(a^2)}{a} \ll (\log L)^2.$$

Proof. We use the standard Titchmarsh divisor theorem for a primitive linear form with nonnegative divisor-bounded multiplicative weights. In the special case needed here it asserts that, for

$$T(x) := \sum_{\substack{a \leq x \\ 4a-1 \text{ prime}}} \tau(a^2),$$

one has

$$T(x) \ll \frac{x}{\log x} \sum_{d \leq x} \frac{2^{\omega(d)}}{\varphi(d)} \quad (x \geq 3). \quad (9)$$

This is the usual switched Titchmarsh theorem for the fixed primitive form $4n - 1$, in the nonnegative divisor-bounded weighted form obtained by dyadic switching and the large sieve; equivalently it is the standard upper-bound Titchmarsh divisor theorem for a primitive linear form, as in Iwaniec–Kowalski [9, Ch. 17]. To indicate the normalization, use the identity

$$\tau(n^2) = \sum_{d|n} 2^{\omega(d)}.$$

It is multiplicative and, on prime powers, reads $2\alpha + 1 = 1 + 2\alpha$. Thus

$$T(x) = \sum_{d \leq x} 2^{\omega(d)} \#\{m \leq x/d : 4dm - 1 \text{ prime}\}.$$

For a fixed d , the primes being counted are the primes $q \leq 4x - 1$ in the single reduced class $q \equiv -1 \pmod{4d}$. Hence the main term has size $x/(\varphi(d) \log x)$, not $x/(d\varphi(d))$. The range $d \leq x^{1/2}(\log x)^{-C}$ is handled by Bombieri–Vinogradov in the progressions $-1 \pmod{4d}$, while the complementary range is treated by the classical Titchmarsh switch in the complementary factor; this gives exactly (9).

It remains to bound the divisor factor. The multiplicative function $a(d) = 2^{\omega(d)}/\varphi(d)$ has local factors

$$\sum_{k \geq 0} a(p^k) = 1 + \sum_{k \geq 1} \frac{2}{p^{k-1}(p-1)} = 1 + \frac{2p}{(p-1)^2} = 1 + \frac{2}{p} + O(p^{-2}).$$

The standard Euler-product upper bound for nonnegative multiplicative functions therefore gives

$$\sum_{d \leq x} \frac{2^{\omega(d)}}{\varphi(d)} \ll (\log x)^2.$$

Combining this with (9) yields

$$T(x) \ll x \log x.$$

Finally, partial summation gives

$$\sum_{\substack{a \leq L \\ 4a-1 \text{ prime}}} \frac{\tau(a^2)}{a} = \frac{T(L)}{L} + \int_1^L \frac{T(t)}{t^2} dt \ll \log L + \int_2^L \frac{\log t}{t} dt \ll (\log L)^2.$$

□

Proposition 4.5 (pairwise-coprime extraction barrier). *Let $\mathcal{B} \subset \{a \leq L\}$ be such that the integers $4a - 1$ with $a \in \mathcal{B}$ are pairwise coprime. Then*

$$\sum_{a \in \mathcal{B}} \frac{|\Lambda_{\text{int}}(a)|}{4a-1} \ll (\log L)^2.$$

Thus no pairwise-coprime extracted subfan can carry the cubic mass in [Proposition 4.2](#).

Proof. Split according as $4a - 1$ is prime or composite. The prime part is bounded by [Lemma 4.4](#), because $|\Lambda_{\text{int}}(a)| \leq \tau(a^2)$ and $(4a - 1)^{-1} \ll a^{-1}$.

For the composite part, put $r = P^-(4a - 1)$. Pairwise coprimality makes these least prime factors distinct. Since $4a - 1$ is composite, $4a - 1 \geq r^2$. For every fixed $\eta_0 > 0$, $|\Lambda_{\text{int}}(a)| \leq \tau(a^2) \ll_{\eta_0} a^{\eta_0}$, and hence

$$\frac{|\Lambda_{\text{int}}(a)|}{4a - 1} \ll_{\eta_0} r^{-2+2\eta_0}.$$

Choosing $\eta_0 < 1/4$ and summing over distinct r gives a bounded composite contribution. The result follows. $\square$

The pairwise-coprime barrier shows that forcing independence costs a logarithmic factor. The construction below instead permits controlled overlaps in the medium coordinate while isolating the large prime factor that drives the dependency estimates.

5 The conductor-separated saturated family

Choose fixed real parameters

$$0 < \eta, \sigma, \lambda, \theta, \beta < 1$$

satisfying

$$3\eta < \lambda, \quad \eta + \sigma < \lambda, \quad \lambda + \eta < \theta, \quad 2\theta < \beta < 1 - \sigma. \quad (10)$$

The first inequality $3\eta < \lambda$ ensures that the dyadic error terms in the progression estimates of [Section 8](#) (worst case $s \approx X^\eta$, $T \approx X^{\lambda-\eta}$) stay negligible against the main term. For example, one may take

$$\eta = \frac{1}{100}, \quad \sigma = \frac{1}{20}, \quad \lambda = \frac{1}{10}, \quad \theta = \frac{3}{10}, \quad \beta = \frac{2}{3}.$$

Put

$$z = (\log X)^4, \quad U = (\log X)^8, \quad S = X^\eta, \quad Y_0 = X^\lambda, \quad Y = X^\sigma, \quad H = X^\theta,$$

and

$$P(z) = \prod_{\ell \leq z} \ell.$$

The product includes the prime 2; all moduli $d_- d_+ p$ below are odd.

Define $\mathcal{E}_{\text{lin}}$ to be the set of integers

$$e = rs^2, \quad \rho(e) = rs,$$

where r and s are squarefree, $(r, s) = 1$, and

$$1 \leq s \leq S, \quad \frac{Y_0}{s} < r \leq \frac{H}{s^2}. \quad (11)$$

Here and throughout the sieve sections e denotes this linearized exact divisor with conductor $\rho(e) = rs$. In the application it is the same object as the divisor $e \mid a^2$ of [Section 2](#); the present notation only fixes the parametrization $e = rs^2$ by the exact square part. Then $e \leq H$, while

$$\rho(e) = rs > Y_0 > Y. \quad (12)$$

This is the conductor separation.

The small-prime saturation is performed after conditioning on a reduced residue class modulo $P(z)$. Fix

$$b \in (\mathbb{Z}/P(z)\mathbb{Z})^\times.$$

For $e \in \mathcal{E}_{\text{lin}}$, choose a small squarefree odd divisor d_- and a rough squarefree cofactor d_+ such that

$$d_- \leq U, \quad d_- \mid P(z), \quad b \equiv -4e \pmod{d_-}, \quad (13)$$

$$d_- d_+ \leq Y, \quad (d_+, P(z)) = 1, \quad (d_- d_+, 4\rho(e)) = 1. \quad (14)$$

Finally choose a prime p satisfying

$$X^\beta < p \leq X^{1-\sigma}, \quad d_- d_+ p \equiv -1 \pmod{4\rho(e)}. \quad (15)$$

For every quadruple $i = (e, d_-, d_+, p)$ satisfying these conditions, define

$$Q_i = d_- d_+ p, \quad q_i = d_+ p,$$

and

$$a_i = \frac{Q_i + 1}{4}, \quad \mathcal{A}_i^{(b)} = \{n : n \equiv -4e \pmod{q_i}\}, \quad w_i^{(b)} = \frac{1}{q_i}.$$

The event $\mathcal{A}_i^{(b)}$ is viewed inside the conditional residue class $n \equiv b \pmod{P(z)}$. Since $d_- \mid P(z)$, the base condition and the small congruence in (13) give $n \equiv -4e \pmod{d_-}$. Combining this with $n \equiv -4e \pmod{d_+p}$ and $(d_-, d_+p) = 1$, any integer in that conditional class and in $\mathcal{A}_i^{(b)}$ satisfies

$$n \equiv -4e \pmod{Q_i}.$$

The congruence (15) implies that a_i is an integer and $\rho(e) \mid a_i$. Because $e = rs^2$ and $\rho(e) = rs$, this gives $e \mid a_i^2$; also $Q_i = 4a_i - 1$. Thus every such hit is a valid certificate by Lemma 2.3. Let $\mathcal{I}_b$ be the set of all quadruples above and set

$$\mu_b = \sum_{i \in \mathcal{I}_b} w_i^{(b)}. \quad (16)$$

Since $d_-d_+ \leq X^\sigma$ and $p \leq X^{1-\sigma}$, both q_i and Q_i are at most X .

In the conditional modulus, the rough part d_+ remains visible, so all residual common medium primes exceed z . The small part d_- is accounted for by the congruence $b \equiv -4e \pmod{d_-}$, contributing its reciprocal average through the exact-divisor coordinate and restoring the logarithmic factor lost in the one-rough-coordinate family.

6 Cited theorems

The proof below uses standard squarefree reciprocal sums, squarefree reciprocal- φ estimates, the Selberg upper-bound sieve, the linear sieve fundamental lemma, maximal Bombieri–Vinogradov, Brun–Titchmarsh, Suen’s correlation inequality, and routine prime and Euler-product estimates. The Titchmarsh divisor theorem is used only in the pairwise-coprime extraction barrier. No shared-kernel estimate from the full divisor fan is used in the proof of Theorem 1.1: after conditioning on $b \pmod{P(z)}$, the sparse family reduces the residual common-prime geometry to large-prime uniqueness.

7 Analytic estimates

All constants in this section may depend on the fixed parameters in (10) and on explicitly displayed auxiliary constants. They are independent of X .

The following convention applies in this section and in Section 8. A bound described as uniform has constants and a threshold X_0 independent of every variable quantified in the statement, including residue classes and auxiliary moduli. These constants may depend only on the fixed parameters in (10) and on auxiliary constants explicitly declared fixed. Likewise, $o(1)$ is uniform when its supremum over all quantified variables tends to zero as $X \rightarrow \infty$.

Lemma 7.1 (Selberg upper bound in long progressions). *Let $A_0 > 3$ and $c_0 > 0$ be fixed. Let D be squarefree with $(D, P(z)) = 1$, and let $(b, D) = 1$. If*

$$U_0 \geq Dz^{A_0}, \quad U_1/U_0 \geq X^{c_0},$$

then

$$\sum_{\substack{U_0 < n \leq U_1 : n \equiv b \pmod{D} \\ (n, P(z)) = 1}} \frac{1}{n} \ll_{A_0, c_0} \frac{1}{D \log z} \log \frac{U_1}{U_0}.$$

The same upper bound remains true after imposing any additional condition $(n, M) = 1$.

References: Selberg upper-bound [7, Sec. 7.1]; linear sieve [7, Thm. 12.20]; see also [8]. Bombieri–Vinogradov: original sources [1, 17]; maximal form used below [9, Thm. 17.1].

Proof. It suffices to prove the corresponding counting estimate on dyadic intervals $(U, 2U] \subset (U_0, U_1]$. Let

$$\mathcal{B} = \{n \in (U, 2U] : n \equiv b \pmod{D}\}.$$

For every squarefree $h \mid P(z)$, the congruences $n \equiv b \pmod{D}$ and $n \equiv 0 \pmod{h}$ combine to one residue class modulo Dh , since $(D, h) = 1$. Thus

$$\#\{n \in \mathcal{B} : h \mid n\} = \frac{U}{Dh} + O(1).$$

The Selberg upper-bound sieve with level z gives

$$\#\{n \in \mathcal{B} : (n, P(z)) = 1\} \ll \frac{U}{D} \prod_{\ell < z} \left(1 - \frac{1}{\ell}\right) + O(z^{2+o(1)}).$$

By Mertens’ theorem the product is $\asymp 1/\log z$. Since $U/D \geq U_0/D \geq z^{A_0}$ and $A_0 > 3$, the remainder $O(z^{2+o(1)})$ is absorbed into the main term. Hence the dyadic count is $\ll U/(D \log z)$. Multiplying by $1/U$ and summing over the $O(\log(U_1/U_0))$ dyadic intervals gives the reciprocal estimate. The extra condition $(n, M) = 1$ can only remove terms. $\square$

Lemma 7.2 (ordinary squarefree reciprocal sums). *Let $0 < a_0 < b_0$ and $c_0 > 0$ be fixed. Uniformly for $X^{a_0} \leq U_0 < U_1 \leq X^{b_0}$ with $U_1/U_0 \geq X^{c_0}$, one has*

$$\sum_{\substack{U_0 < n \leq U_1 \\ n \text{ squarefree}}} \frac{1}{n} \asymp_{a_0, b_0, c_0} \log \frac{U_1}{U_0}.$$

If M is squarefree and $M \leq X^{C_0}$ for a fixed C_0 , then

$$\sum_{\substack{U_0 < n \leq U_1 \\ n \text{ squarefree} \\ (n, M) = 1}} \frac{1}{n} \asymp_{a_0, b_0, c_0, C_0} \frac{\varphi(M)}{M} \log \frac{U_1}{U_0}. \quad (17)$$

Moreover, for every fixed $C_1 > 0$, every squarefree $D \leq (\log X)^{C_1}$, every reduced class $b \pmod{D}$, and every squarefree $M \leq X^{C_0}$ with $(M, D) = 1$,

$$\sum_{\substack{U_0 < n \leq U_1 \\ n \equiv b \pmod{D} \\ n \text{ squarefree} \\ (n, M) = 1}} \frac{1}{n} \ll_{a_0, b_0, c_0, C_0, C_1} \frac{1}{D} \frac{\varphi(M)}{M} \log \frac{U_1}{U_0}. \quad (18)$$

Finally, for every squarefree $D \leq X^{b_0}$ and every reduced class $b \pmod{D}$,

$$\sum_{\substack{U_0 < n \leq U_1 \\ n \equiv b \pmod{D}}} \frac{1}{n} \ll \frac{1}{D} \log \frac{U_1}{U_0} + \frac{1}{U_0}. \quad (19)$$

Proof. We first count on dyadic intervals. Put $\mathcal{I} = (T, 2T]$, where $X^{a_0} \leq T \leq X^{b_0}$, and let M be squarefree. By Mobius inversion for squarefreeness,

$$\#\{n \in \mathcal{I} : n \text{ squarefree}, (n, M) = 1\} = \sum_{\substack{a \leq (2T)^{1/2} \\ (a, M) = 1}} \mu(a) \#\{m : T/a^2 < m \leq 2T/a^2, (m, M) = 1\}.$$

For any interval of length Y , the number of integers coprime to squarefree M is $Y\varphi(M)/M + O(\tau(M))$. Thus the last display equals

$$T \frac{\varphi(M)}{M} \sum_{\substack{a \leq (2T)^{1/2} \\ (a, M) = 1}} \frac{\mu(a)}{a^2} + O(T^{1/2} \tau(M)).$$

The truncated Euler product is

$$\sum_{\substack{a \leq (2T)^{1/2} \\ (a, M) = 1}} \frac{\mu(a)}{a^2} = \frac{1}{\zeta(2)} \prod_{p|M} \left(1 - \frac{1}{p^2}\right)^{-1} + O(T^{-1/2}),$$

and the product over $p \mid M$ is bounded above and below by absolute positive constants. Since $M \leq X^{C_0}$, $\tau(M) = X^{o(1)}$ and $\varphi(M)/M \gg_{C_0} 1/\log X$; hence the error is negligible compared with $T\varphi(M)/M$. We obtain

$$\#\{T < n \leq 2T : n \text{ squarefree}, (n, M) = 1\} \asymp_{a_0, b_0, C_0} T \frac{\varphi(M)}{M}. \quad (20)$$

Partial summation proves (17); taking $M = 1$ gives the first assertion.

The progression estimate (18) is proved by the same signed inclusion with one extra congruence. On $(T, 2T]$, expand with $u \mid M$ and $v^2 \mid n$. The conditions $n \equiv b \pmod{D}$, $u \mid n$, and $v^2 \mid n$ are soluble only when $(uv, D) = 1$; in that case they combine into one residue class modulo $D \operatorname{lcm}(u, v^2)$. Hence the dyadic count is

$$TC(M, D) + O(\tau(M)T^{1/2}), \quad C(M, D) = \frac{1}{D} \sum_{u|M} \mu(u) \sum_{\substack{v \geq 1 \\ (v, D) = 1}} \frac{\mu(v)}{\operatorname{lcm}(u, v^2)}.$$

The Euler product for $C(M, D)$ has local factor $1 - 1/p$ at primes $p \mid M$, local factor $1 - 1/p^2$ at primes $p \nmid MD$, and local factor 1 at primes $p \mid D$. Therefore

$$C(M, D) = \frac{1}{D} \frac{1}{\zeta(2)} \prod_{p \mid D} \left(1 - \frac{1}{p^2}\right)^{-1} \prod_{p \mid M} \frac{1 - 1/p}{1 - 1/p^2} \asymp \frac{1}{D} \frac{\varphi(M)}{M}.$$

The error is negligible in the polynomial range because $D \leq (\log X)^{C_1}$. Thus the dyadic count is $\ll T\varphi(M)/(DM)$. Partial summation proves (18). The final progression upper bound (19) follows by dropping all arithmetic conditions except the progression and summing $1/(b + Dm)$. $\square$

Lemma 7.3 (average squarefree conductor mass). *For $S = X^\eta$ one has*

$$\sum_{\substack{s \leq S \\ s \text{ squarefree}}} \frac{\varphi(s)}{s^2} \asymp \log S, \quad \sum_{\substack{s \leq S \\ s \text{ squarefree}}} \frac{1}{s} \asymp \log S.$$

Proof. For the second estimate,

$$\sum_{s \geq 1} \frac{\mu^2(s)}{s^{1+w}} = \frac{\zeta(1+w)}{\zeta(2+2w)},$$

which has a simple pole at $w = 0$. For the first, the Dirichlet series factors as

$$\sum_{s \geq 1} \frac{\mu^2(s)\varphi(s)}{s^2} s^{-w} = \prod_p \left(1 + \frac{p-1}{p^{2+w}}\right),$$

whose local factor is $1 + p^{-1} + O(p^{-2})$, giving a simple pole at $w = 0$. A Tauberian theorem, or partial summation from the matching counting estimate, supplies the logarithmic order. $\square$

Lemma 7.4 (dyadic and reciprocal rough squarefree estimates). *Let $0 < a_0 < b_0$ and $C_0 > 0$ be fixed. Uniformly for $M \leq X^{C_0}$ and $X^{a_0} \leq t \leq X^{b_0}$,*

$$\#\{t < n \leq 2t : n \text{ squarefree}, (n, P(z)) = 1, (n, M) = 1\} \asymp_{a_0, b_0, C_0} \frac{t}{\log z}. \quad (21)$$

Consequently, if $a_0 \leq a < b \leq b_0$ and $b - a \geq c_1 > 0$, then, uniformly in a, b and $M \leq X^{C_0}$,

$$\sum_{\substack{X^a < n \leq X^b \\ n \text{ squarefree} \\ (n, P(z)) = 1 \\ (n, M) = 1}} \frac{1}{n} \asymp_{a_0, b_0, c_1, C_0} \frac{\log X}{\log z}. \quad (22)$$

Proof. We first prove (21). Put $\mathcal{A} = (t, 2t] \cap \mathbb{Z}$ and $P = P(z)$. For each squarefree $h \mid P$,

$$A_h := \#\{n \in \mathcal{A} : h \mid n\} = \frac{t}{h} + O(1).$$

Apply the lower- and upper-bound linear sieve in dimension one with level $D_0 = t^{1/3}$. The sieve parameter is

$$s_0 = \frac{\log D_0}{\log z} = \frac{(1/3) \log t}{4 \log \log X} \geq \frac{a_0}{12} \frac{\log X}{\log \log X} \rightarrow \infty$$

uniformly in the stated range. The fundamental lemma therefore gives

$$\#\{t < n \leq 2t : (n, P(z)) = 1\} = t \prod_{\ell < z} \left(1 - \frac{1}{\ell}\right) \{1 + O(e^{-cs_0})\} + O(D_0^{1+o(1)}), \quad (23)$$

with an absolute $c > 0$. Since $D_0^{1+o(1)} = o(t/\log z)$ and Mertens' theorem gives $\prod_{\ell < z} (1 - 1/\ell) \asymp 1/\log z$, this is $\asymp t/\log z$, with both upper and lower bounds supplied by the same linear-sieve estimate.

The squarefree condition removes only a negligible proportion of the rough integers counted in (23). If a rough integer in $(t, 2t]$ is not squarefree, then it is divisible by ℓ^2 for some prime $\ell \geq z$. Hence the number of such integers is at most

$$\sum_{z \leq \ell \leq \sqrt{2t}} \left(\frac{t}{\ell^2} + 1\right) \ll \frac{t}{z \log z} + \frac{\sqrt{t}}{\log t} = o\left(\frac{t}{\log z}\right),$$

uniformly in $t \geq X^{a_0}$.

It remains to impose $(n, M) = 1$. Prime factors of M below z are already excluded. For the remaining prime factors,

$$\sum_{\substack{\ell \mid M \\ \ell \geq z}} \frac{1}{\ell} \leq \frac{\log M}{z \log z} \ll_{C_0} \frac{\log X}{z \log z} = o\left(\frac{1}{\log z}\right). \quad (24)$$

The inequality follows by maximizing the reciprocal sum subject to the constraint that the product of the selected primes is at most M : the largest contribution comes from primes just above z . By the union bound, the number of integers in $(t, 2t]$ divisible by at least one prime $\ell \mid M$, $\ell \geq z$, is at most

$$\sum_{\substack{\ell \mid M \\ z \leq \ell \leq 2t}} \left(\frac{t}{\ell} + 1 \right) \ll t \sum_{\substack{\ell \mid M \\ \ell \geq z}} \frac{1}{\ell} + \omega(M) = o\left(\frac{t}{\log z}\right),$$

since $\omega(M) \ll_{C_0} \log X / \log z = o(t / \log z)$. This proves the dyadic count (21).

For the reciprocal estimate, decompose $(X^a, X^b]$ into dyadic intervals, using one shortened interval at each endpoint if necessary. On each interval $(u, v]$ with $v \leq 2u$, the dyadic count gives a contribution $\asymp (v - u) / (u \log z)$ when the interval has fixed relative length; the endpoint pieces are handled by splitting once more into boundedly many such intervals. Equivalently, partial summation applied to the counting function supplied by (21) gives

$$\sum_{\substack{X^a < n \leq X^b \\ n \text{ squarefree} \\ (n, P(z))=1 \\ (n, M)=1}} \frac{1}{n} \asymp \frac{1}{\log z} \int_{X^a}^{X^b} \frac{du}{u} \asymp_{c_1} \frac{\log X}{\log z},$$

which proves (22). □

Lemma 7.5 (initial-segment rough reciprocal sums). *Let $\sigma_0 > 0$ and $C_0 > 0$ be fixed. Uniformly for $M \leq X^{C_0}$,*

$$\sum_{\substack{n \leq X^{\sigma_0} \\ n \text{ squarefree} \\ (n, P(z))=1 \\ (n, M)=1}} \frac{1}{n} \asymp_{\sigma_0, C_0} \frac{\log X}{\log z}.$$

Proof. For the lower bound, keep only $X^{\sigma_0/2} < n \leq X^{\sigma_0}$ and apply Lemma 7.4. For the upper bound, put $U = z^4$. The initial segment is bounded by

$$\sum_{n \leq U} \frac{1}{n} \ll 1 + \log U \ll \log z \ll \frac{\log X}{\log z}.$$

For large X we have $U \leq X^{\sigma_0/2}$, so $X^{\sigma_0}/U \geq X^{\sigma_0/3}$. Apply Lemma 7.1 to $(U, X^{\sigma_0}]$ with $D = 1$, imposing $(n, M) = 1$ as the additional condition. It gives

$$\sum_{\substack{U < n \leq X^{\sigma_0} \\ (n, P(z))=1 \\ (n, M)=1}} \frac{1}{n} \ll_{\sigma_0} \frac{\log X}{\log z}.$$

Adding the two ranges proves the upper bound. □

Theorem 7.6 (Bombieri–Vinogradov, maximal form). *Let $A > 0$ and $\varepsilon > 0$ be fixed. For $x \geq 3$,*

$$\sum_{q \leq x^{1/2-\varepsilon}} \max_{(a, q)=1} \max_{2 \leq y \leq x} \left| \vartheta(y; q, a) - \frac{y}{\varphi(q)} \right| \ll_{A, \varepsilon} \frac{x}{(\log x)^A}.$$

Here $\vartheta(y; q, a) = \sum_{\substack{p \leq y \\ p \equiv a \pmod{q}}} \log p$.

Proof. This is the standard maximal Bombieri–Vinogradov theorem, originating in the work of Bombieri and Vinogradov [1, 17]; see, for example, Iwaniec and Kowalski [9, Thm. 17.1]. That theorem is commonly stated for $\psi(y; q, a)$ and for moduli $q \leq x^{1/2}/(\log x)^A$. The present weaker range $q \leq x^{1/2-\varepsilon}$ is immediate after increasing the logarithmic saving. The replacement of ψ by ϑ costs only the contribution of prime powers, $O(x^{1-\varepsilon} \log^2 x)$ after summing over moduli in this range, which is absorbed by the displayed bound after increasing the logarithmic saving. The maximal form in y is part of the standard statement. □

Lemma 7.7 (a divisor moment with $1/\varphi$). *For every fixed $K \geq 0$ and $\nu > 0$,*

$$\sum_{m \leq X^\nu} \frac{\tau(m)^K}{\varphi(m)} \ll_{K, \nu} (\log X)^{O_K(1)}.$$

Proof. The case $K = 0$ is included. For every m ,

$$\frac{m}{\varphi(m)} = \prod_{\ell|m} \left(1 - \frac{1}{\ell}\right)^{-1} \leq 2^{\omega(m)} \leq \tau(m).$$

Hence

$$\frac{\tau(m)^K}{\varphi(m)} \leq \frac{\tau(m)^{K+1}}{m}.$$

Let $L = \lceil K + 1 \rceil$. Since $\tau(m) \geq 1$,

$$\sum_{m \leq X^\nu} \frac{\tau(m)^K}{\varphi(m)} \leq \sum_{m \leq X^\nu} \frac{\tau(m)^L}{m} \leq \prod_{\ell \leq X^\nu} \left(\sum_{a \geq 0} \frac{(a+1)^L}{\ell^a} \right).$$

For each prime ℓ the local factor is $1 + 2^L/\ell + O_L(\ell^{-2})$. Therefore the product is

$$\ll_L \exp \left\{ 2^L \sum_{\ell \leq X^\nu} \frac{1}{\ell} + O_L(1) \right\} \ll_{K,\nu} (\log X)^{2^L},$$

which proves the lemma. $\square$

Lemma 7.8 (Brun–Titchmarsh reciprocal bound). *Uniformly for $m \leq X^\nu$ with $\nu < \beta$, and for reduced residue classes $a \pmod{m}$,*

$$\sum_{\substack{X^\beta < p \leq X^{1-\sigma} \\ p \equiv a \pmod{m}}} \frac{1}{p} \ll_{\nu,\beta,\sigma} \frac{1}{\varphi(m)}.$$

Proof. Decompose into dyadic intervals $T < p \leq 2T$. Since $m \leq X^\nu$ and $T \geq X^\beta/2$, the Brun–Titchmarsh theorem gives

$$\#\{T < p \leq 2T : p \equiv a \pmod{m}\} \ll \frac{T}{\varphi(m) \log(T/m)} \ll \frac{T}{\varphi(m) \log X}.$$

Multiplying by $1/T$ and summing over $O(\log X)$ dyadic intervals proves the claim. $\square$

Lemma 7.9 (reciprocal Bombieri–Vinogradov with divisor weights). *Let $A, C > 0$ be fixed, and let $0 < \nu < \beta/2$. Then*

$$\sum_{m \leq X^\nu} \tau(m)^C \max_{(a,m)=1} \left| \sum_{\substack{X^\beta < p \leq X^{1-\sigma} \\ p \equiv a \pmod{m}}} \frac{1}{p} - \frac{1}{\varphi(m)} \sum_{X^\beta < p \leq X^{1-\sigma}} \frac{1}{p} \right| \ll_{A,C} (\log X)^{-A}.$$

Proof. First prove the unweighted estimate with an arbitrary logarithmic saving. On a dyadic interval $T < p \leq 2T$, with $T \geq X^\beta/2$, choose $\varepsilon > 0$ so small that $\nu < \beta(1/2 - \varepsilon)$. Then $X^\nu \leq T^{1/2-\varepsilon}$ for all such T and large X . Applying [Theorem 7.6](#) with $x = 2T$ gives, for every $A_1 > 0$,

$$\sum_{m \leq X^\nu} \max_{(a,m)=1} \max_{T \leq u \leq 2T} \left| \vartheta(u; m, a) - \frac{u}{\varphi(m)} \right| \ll_{A_1, \varepsilon} \frac{T}{(\log T)^{A_1}}. \quad (25)$$

By the prime number theorem with a classical zero-free-region error term and [Lemma 7.7](#) with $K = 0$, replacing $u/\varphi(m)$ by $\vartheta(u)/\varphi(m)$ changes the left side by $O_A(T/(\log T)^A)$ for every fixed A .

Put

$$E(u; m, a) = \vartheta(u; m, a) - \frac{\vartheta(u)}{\varphi(m)}.$$

Stieltjes partial summation with

$$f(u) = \frac{1}{u \log u}$$

gives the exact identity

$$\begin{aligned} & \sum_{\substack{T < p \leq 2T \\ p \equiv a \pmod{m}}} \frac{1}{p} - \frac{1}{\varphi(m)} \sum_{T < p \leq 2T} \frac{1}{p} \\ &= f(2T)E(2T; m, a) - f(T)E(T; m, a) - \int_T^{2T} E(u; m, a) f'(u) du. \end{aligned}$$

Since $|f'(u)| \ll 1/(u^2 \log u)$ on $[T, 2T]$, (25) implies

$$\sum_{m \leq X^\nu} \max_{(a,m)=1} \left| \sum_{\substack{T < p \leq 2T \\ p \equiv a \pmod{m}}} \frac{1}{p} - \frac{1}{\varphi(m)} \sum_{T < p \leq 2T} \frac{1}{p} \right| \ll_{A_1} \frac{1}{(\log T)^{A_1+1}}.$$

Summing over $O(\log X)$ dyadic intervals gives the unweighted reciprocal Bombieri–Vinogradov estimate with any prescribed logarithmic saving.

Now insert the factor $\tau(m)^C$. Let G_m denote the maximum reciprocal-prime error over the full interval $X^\beta < p \leq X^{1-\sigma}$. By Lemma 7.8,

$$G_m \ll \frac{1}{\varphi(m)} \quad (m \leq X^\nu). \quad (26)$$

Choose the constants in the following order. First fix $h = 1$. Let $C_0 = C_0(C)$ be the exponent supplied by Lemma 7.7 for the moment $\tau(m)^{2C}$, and choose $D > A + C_0 + 5$; finally in the unweighted reciprocal Bombieri–Vinogradov estimate choose the saving $A_1 = A + D + 5$. Put $L_0 = (\log X)^D$ and split the moduli according as $\tau(m)^C \leq L_0$. For the low-divisor part,

$$\sum_{\substack{m \leq X^\nu \\ \tau(m)^C \leq L_0}} \tau(m)^C G_m \leq L_0 \sum_{m \leq X^\nu} G_m \ll (\log X)^D (\log X)^{-A-D-5} \ll (\log X)^{-A}.$$

For the high-divisor part, $\tau(m)^C > L_0$ implies $\tau(m)^C \leq L_0^{-h} \tau(m)^{C(1+h)}$. Hence (26) and Lemma 7.7 give

$$\sum_{\substack{m \leq X^\nu \\ \tau(m)^C > L_0}} \tau(m)^C G_m \ll L_0^{-h} \sum_{m \leq X^\nu} \frac{\tau(m)^{C(1+h)}}{\varphi(m)} \ll (\log X)^{-Dh+C_0} \ll (\log X)^{-A}.$$

Combining the low- and high-divisor ranges proves the lemma. $\square$

8 Saturated mass and endpoint-safe tensorisation

The estimates in this section are uniform in the fixed reduced class $b \pmod{P(z)}$ from Section 5.

Proposition 8.1 (quadratic exact-divisor mass). *The linearized exact-divisor family, before any sieving at small primes, satisfies*

$$M_1 := \sum_{e \in \mathcal{E}_{\text{lin}}} \frac{1}{\rho(e)} \asymp (\log X)^2, \quad M_\varphi := \sum_{e \in \mathcal{E}_{\text{lin}}} \frac{1}{\varphi(\rho(e))} \asymp (\log X)^2.$$

Proof. Write $e = rs^2$, so that $\rho(e) = rs$. For fixed admissible s , the variable r lies in $Y_0/s < r \leq H/s^2$, and we have logarithmic length at least $(\theta - \lambda - \eta) \log X$. By Lemma 7.2,

$$\sum_{\substack{Y_0/s < r \leq H/s^2 \\ r \text{ squarefree} \\ (r,s)=1}} \frac{1}{r} \asymp \frac{\varphi(s)}{s} \log \frac{H}{Y_0 s},$$

uniformly for $s \leq S$. Since

$$(\theta - \lambda - \eta) \log X \leq \log \frac{H}{Y_0 s} \leq (\theta - \lambda) \log X,$$

this logarithmic length is comparable to $\log X$ throughout the s -range. Thus

$$M_1 \asymp \sum_{\substack{s \leq S \\ s \text{ squarefree}}} \frac{\varphi(s)}{s^2} \log \frac{H}{Y_0 s} \asymp (\log X)^2$$

by Lemma 7.3.

We next prove the corresponding estimate with $1/\varphi(r)$ in a form that keeps its dependence on s explicit. For squarefree r ,

$$\frac{1}{\varphi(r)} = \frac{1}{r} \sum_{k|r} \gamma(k), \quad \gamma(k) = \mu^2(k) \prod_{p|k} \frac{1}{p-1}.$$

The elementary estimates

$$\sum_{k \leq K} \gamma(k) \ll \log(2K), \quad \sum_{k \geq 1} \frac{\gamma(k)}{k} < \infty, \quad \sum_{k > K} \frac{\gamma(k)}{k} \ll \frac{\log(2K)}{K} \quad (27)$$

will be used repeatedly. The first two follow from the corresponding Euler products, and the last follows from the first by partial summation. Fix $0 < \kappa < \lambda - \eta$ and put $K_0 = X^\kappa$. The lower bound follows from $1/\varphi(r) \geq 1/r$ and the preceding reciprocal estimate. For the upper bound, interchange the r - and k -sums. When $k \leq K_0$, write $r = kt$. Only squarefree k coprime to s occur, and t is squarefree with $(t, sk) = 1$. Since

$$\frac{Y_0}{sk} \geq X^{\lambda-\eta-\kappa}, \quad sk \leq X^{\eta+\kappa},$$

[Lemma 7.2](#) applies uniformly and gives

$$\begin{aligned} & \sum_{\substack{Y_0/s < r \leq H/s^2 \\ r \text{ squarefree} \\ (r,s)=1, k|r}} \frac{1}{r} \\ &= \frac{1}{k} \sum_{\substack{Y_0/(sk) < t \leq H/(s^2k) \\ t \text{ squarefree} \\ (t,sk)=1}} \frac{1}{t} \ll \frac{1}{k} \frac{\varphi(s)}{s} \log \frac{H}{Y_0 s}. \end{aligned}$$

Here we used $\varphi(sk)/(sk) \leq \varphi(s)/s$. Multiplication by $\gamma(k)$ and summation over $k \leq K_0$ costs only an absolute constant, by (27).

For $k > K_0$, discard squarefreeness and coprimality. The inner harmonic sum after writing $r = kt$ is at most $1 + \log(H/s^2)$, which is $O(\log(H/(Y_0 s)))$ uniformly in s . By (27),

$$\sum_{k > K_0} \frac{\gamma(k)}{k} \ll \frac{\log K_0}{K_0} = o\left(\frac{\varphi(s)}{s}\right)$$

uniformly for $s \leq S$. Here $s/\varphi(s) \leq \tau(s) = X^{o(1)}$, so $(\log K_0)/K_0 = o(\varphi(s)/s)$ throughout this range. Consequently

$$\sum_{\substack{Y_0/s < r \leq H/s^2 \\ r \text{ squarefree} \\ (r,s)=1}} \frac{1}{\varphi(r)} \asymp \frac{\varphi(s)}{s} \log \frac{H}{Y_0 s}.$$

Finally $\varphi(rs) = \varphi(r)\varphi(s)$, so

$$M_\varphi \asymp \sum_{\substack{s \leq S \\ s \text{ squarefree}}} \frac{1}{\varphi(s)} \frac{\varphi(s)}{s} \log \frac{H}{Y_0 s} \asymp \sum_{\substack{s \leq S \\ s \text{ squarefree}}} \frac{1}{s} \log \frac{H}{Y_0 s} \asymp (\log X)^2,$$

again by [Lemma 7.3](#), after carrying out the partial summation over the slanted region $Y_0/s < r \leq H/s^2$. □

Theorem 8.2 (endpoint-safe exact-divisor upper tensorisation). *Let $D \leq YU$ be squarefree and let $(c, D) = 1$. Then*

$$\sum_{\substack{e \in \mathcal{E}_{\text{fin}} \\ e \equiv c \pmod{D}}} \frac{1}{\rho(e)} \ll \frac{M_1}{D}, \quad \sum_{\substack{e \in \mathcal{E}_{\text{fin}} \\ e \equiv c \pmod{D}}} \frac{1}{\varphi(\rho(e))} \ll \frac{M_\varphi}{D}.$$

Proof. Write $e = rs^2$. If $(s, D) > 1$, then the congruence $rs^2 \equiv c \pmod{D}$ has no solution, so we may assume $(s, D) = 1$. It then reduces to the single reduced class $r \equiv cs^{-2} \pmod{D}$. With $U_0 = Y_0/s$ and $U_1 = H/s^2$ we have

$$U_0/D \geq X^{\lambda-\eta-\sigma} (\log X)^{-8} \rightarrow \infty, \quad U_1/U_0 \geq X^{\theta-\lambda-\eta}.$$

The elementary progression bound gives the endpoint-sensitive estimate

$$\sum_{\substack{U_0 < r \leq U_1 \\ r \equiv cs^{-2} \pmod{D}}} \frac{1}{r} \ll \frac{1}{D} \log \frac{U_1}{U_0} + \frac{1}{U_0}.$$

The endpoint term is negligible uniformly: since $D \leq YU = X^{\sigma+o(1)}$, $U_0 \geq X^{\lambda-\eta}$, and $\lambda > \eta + \sigma$, we have $U_0^{-1} = o(D^{-1} \log(U_1/U_0))$. Hence the displayed estimate is $\ll D^{-1} \log(U_1/U_0)$ uniformly. For squarefree r ,

$$\frac{1}{\varphi(r)} = \frac{1}{r} \sum_{k|r} \gamma(k), \quad \gamma(k) = \mu^2(k) \prod_{\ell|k} \frac{1}{\ell-1}.$$

Choose κ with $\sigma < \kappa < \lambda - \eta$. Terms with $(k, D) > 1$ vanish, since $r \equiv cs^{-2} \pmod{D}$ is a unit. For $k \leq X^\kappa$ write $r = kt$; then $t \equiv cs^{-2}k^{-1} \pmod{D}$, and the same endpoint-sensitive progression estimate now reads

$$\sum_{\substack{U_0 < r \leq U_1 \\ r \equiv cs^{-2} \pmod{D} \\ k|r}} \frac{1}{r} \ll \frac{1}{kD} \log \frac{U_1}{U_0} + \frac{1}{U_0}. \quad (28)$$

After writing $r = kt$, the first term is the expected progression contribution, while a possible incomplete initial progression contributes at most $1/U_0$ in the original r -variable. By (27), the endpoint part in (28), after multiplication by $\gamma(k)$ and summation over $k \leq X^\kappa$, is

$$\ll \frac{\log X}{U_0} \leq X^{-\lambda+\eta+o(1)} \log X = o\left(\frac{1}{D} \log \frac{U_1}{U_0}\right),$$

because $D \leq YU = X^{\sigma+o(1)}$ and $\lambda > \eta + \sigma$. The main part in (28) contributes

$$\ll \frac{1}{D} \log \frac{U_1}{U_0} \sum_{k \leq X^\kappa} \frac{\gamma(k)}{k} \ll \frac{1}{D} \log \frac{U_1}{U_0},$$

again by (27). For $k > X^\kappa$ we discard the congruence and all coprimality conditions. After writing $r = kt$, the remaining harmonic sum is $O(\log(U_1/U_0))$: both $\log U_1$ and $\log(U_1/U_0)$ are bounded above and below by fixed positive multiples of $\log X$. Hence this range contributes

$$\ll \log \frac{U_1}{U_0} \sum_{k > X^\kappa} \frac{\gamma(k)}{k} \ll \log \frac{U_1}{U_0} \frac{\log X}{X^\kappa} = o\left(\frac{1}{D} \log \frac{U_1}{U_0}\right),$$

because $D \leq X^{\sigma+o(1)}$ and $\kappa > \sigma$. We have proved, uniformly in s, D, c ,

$$\sum_{\substack{U_0 < r \leq U_1 \\ r \text{ squarefree} \\ (r,s)=1 \\ r \equiv cs^{-2} \pmod{D}}} \frac{1}{\varphi(r)} \ll \frac{1}{D} \log \frac{U_1}{U_0}.$$

The same divisor expansion also gives

$$\sum_{\substack{s \leq S \\ s \text{ squarefree}}} \frac{1}{\varphi(s)} \leq \sum_{k \leq S} \frac{\gamma(k)}{k} \sum_{t \leq S/k} \frac{1}{t} \ll \log S.$$

Multiplying the fixed- s estimate by $1/\varphi(s)$ and summing over s therefore gives $\ll D^{-1}(\log X)^2 \ll M_\varphi/D$, by [Proposition 8.1](#). The first estimate follows in the same way from the preceding $1/r$ bound after multiplication by $1/s$, using $\sum_{s \leq S} \mu^2(s)/s \ll \log S$ and $M_1 \asymp (\log X)^2$. $\square$

The saturation lower bound is an averaged small-prime statement: the small-prime factors are summed before the lower bound is taken, so the possible loss from a fixed small modulus is balanced by the average over d .

Lemma 8.3 (reciprocal- φ progression average). *Let $d \leq U$ be squarefree and odd, let $(a, d) = 1$, and let $s \leq S$ be squarefree with $(s, d) = 1$. Put $U_0 = Y_0/s$ and $U_1 = H/s^2$. Uniformly in all these parameters,*

$$\sum_{\substack{U_0 < r \leq U_1 \\ r \text{ squarefree} \\ (r,s)=1 \\ r \equiv a \pmod{d}}} \frac{1}{\varphi(r)} \asymp \frac{1}{d} \frac{\varphi(s)}{s} \log \frac{U_1}{U_0}.$$

Proof. We first prove a dyadic counting estimate. Let $U_0 \leq T \leq U_1/2$, and recall that, since $(a, d) = 1$, every $r \equiv a \pmod{d}$ is coprime to d . By Möbius inversion,

$$\mathbf{1}_{r \text{ squarefree}} \mathbf{1}_{(r,s)=1} = \sum_{v^2|r} \mu(v) \sum_{u|(r,s)} \mu(u).$$

The congruences $r \equiv a \pmod{d}$, $u \mid r$, and $v^2 \mid r$ are soluble only when $(uv, d) = 1$; then they combine into one class modulo $d \operatorname{lcm}(u, v^2)$. Therefore

$$\begin{aligned} & \#\{T < r \leq 2T : r \equiv a \pmod{d}, r \text{ squarefree}, (r, s) = 1\} \\ &= TC(s, d) + O(\tau(s)T^{1/2}), \end{aligned}$$

where

$$C(s, d) = \frac{1}{d} \sum_{u \mid s} \mu(u) \sum_{\substack{v \geq 1 \\ (v, d)=1}} \frac{\mu(v)}{\operatorname{lcm}(u, v^2)}.$$

The Euler product for $C(s, d)$ is explicit: for $p \nmid d$ the local factor is $1 - 1/p$ if $p \mid s$ and $1 - 1/p^2$ if $p \nmid s$, while primes $p \mid d$ contribute 1. Hence

$$C(s, d) = \frac{1}{d} \frac{1}{\zeta(2)} \prod_{p \mid d} \left(1 - \frac{1}{p^2}\right)^{-1} \prod_{p \nmid s} \frac{1 - 1/p}{1 - 1/p^2} \asymp \frac{1}{d} \frac{\varphi(s)}{s}.$$

Here the products remaining after extracting $\varphi(s)/s$ are bounded by absolute constants. To include both endpoints, let

$$N_{s,d,a}(x) = \#\{r \leq x : r \equiv a \pmod{d}, r \text{ squarefree}, (r, s) = 1\}.$$

The same inclusion–exclusion calculation on $[1, x]$ gives

$$N_{s,d,a}(x) = C(s, d)x + O(\tau(s)x^{1/2}) \quad (x \geq U_0),$$

uniformly in a, d, s . Partial summation on $(U_0, U_1]$ therefore yields

$$\sum_{\substack{U_0 < r \leq U_1 \\ r \equiv a \pmod{d} \\ r \text{ squarefree} \\ (r, s)=1}} \frac{1}{r} = C(s, d) \log \frac{U_1}{U_0} + O\left(\frac{\tau(s)}{\sqrt{U_0}}\right).$$

The ratio of the error to the main term is

$$\ll \frac{d \tau(s) s}{\varphi(s) \sqrt{U_0} \log(U_1/U_0)} = o(1)$$

uniformly, because $U_0 \geq X^{\lambda-\eta}$, $d \leq (\log X)^8$, and $\log(U_1/U_0) \gg \log X$. Hence

$$\sum_{\substack{U_0 < r \leq U_1 \\ r \text{ squarefree} \\ (r, s)=1 \\ r \equiv a \pmod{d}}} \frac{1}{r} \asymp \frac{1}{d} \frac{\varphi(s)}{s} \log \frac{U_1}{U_0}. \quad (29)$$

Since $1/\varphi(r) \geq 1/r$, this is the lower bound.

For the upper bound use, for squarefree r , the divisor expansion

$$\frac{1}{\varphi(r)} = \frac{1}{r} \sum_{k \mid r} \gamma(k), \quad \gamma(k) = \mu^2(k) \prod_{p \mid k} \frac{1}{p-1}.$$

Choose a fixed $\kappa > 0$ with $\eta + \kappa < \lambda$ and split at $K_0 = X^\kappa$. Terms with $(k, d) > 1$ give no contribution, since $r \equiv a \pmod{d}$ is a unit; hence we may assume $(k, d) = 1$. For $k \leq K_0$ we write $r = kt$. Then t is squarefree, $(t, sk) = 1$, and $t \equiv ak^{-1} \pmod{d}$. The interval for t has lower endpoint at least $X^{\lambda-\eta-\kappa}$ and the same logarithmic length as the interval for r . Applying (18) with $M = sk$ gives

$$\sum_{\substack{U_0/k < t \leq U_1/k \\ t \equiv ak^{-1} \pmod{d} \\ t \text{ squarefree}, (t, sk)=1}} \frac{1}{kt} \ll \frac{1}{k} \frac{1}{d} \frac{\varphi(sk)}{sk} \log \frac{U_1}{U_0} \leq \frac{1}{k} \frac{1}{d} \frac{\varphi(s)}{s} \log \frac{U_1}{U_0}.$$

After multiplication by $\gamma(k)$ and summation over $k \leq K_0$, this is

$$\ll \frac{1}{d} \frac{\varphi(s)}{s} \log \frac{U_1}{U_0},$$

by (27).

For $k > K_0$ we drop all congruence and coprimality conditions. After writing $r = kt$, the remaining harmonic sum is $O(\log(U_1/U_0))$ uniformly, because $\log U_1 \ll \log X$ whereas $\log(U_1/U_0) \gg \log X$. Since $\gamma(k) \leq 2^{\omega(k)}/k$ for squarefree k ,

$$\sum_{k > K_0} \frac{\gamma(k)}{k} \ll \frac{\log K_0}{K_0}$$

by (27). Thus the large- k contribution is

$$\ll \log \frac{U_1}{U_0} \sum_{k > K_0} \frac{\gamma(k)}{k} = o\left(\frac{1}{d} \frac{\varphi(s)}{s} \log \frac{U_1}{U_0}\right),$$

since

$$d \frac{s}{\varphi(s)} \frac{\log K_0}{K_0} \leq X^{-\kappa+o(1)} (\log X)^9 = o(1).$$

This gives the upper bound with a threshold independent of a, d, s . □

Lemma 8.4 (averaged small-prime divisor sum). *For*

$$A_s := \sum_{\substack{d \leq U \\ d|P(z) \\ d \text{ odd} \\ (d,s)=1}} \frac{1}{d}$$

one has

$$\sum_{\substack{s \leq S \\ s \text{ squarefree}}} \frac{A_s}{s} \asymp \log S \log z.$$

Proof. The upper bound follows at once from $A_s \ll \log z$ and $\sum_{s \leq S} \mu^2(s)/s \ll \log S$. For the lower bound, we interchange the sums:

$$\sum_{s \leq S} \frac{\mu^2(s) A_s}{s} = \sum_{\substack{d \leq U \\ d|P(z) \\ d \text{ odd}}} \frac{1}{d} \sum_{\substack{s \leq S \\ s \text{ squarefree} \\ (s,d)=1}} \frac{1}{s}.$$

For squarefree $d \leq U$, the needed estimate follows from Möbius inversion. For $T \geq 2$, it gives

$$Q_d(T) := \sum_{\substack{n \leq T \\ n \text{ squarefree} \\ (n,d)=1}} \frac{1}{n} = \sum_{\substack{k \leq \sqrt{T} \\ (k,d)=1}} \frac{\mu(k)}{k^2} \sum_{\substack{m \leq T/k^2 \\ (m,d)=1}} \frac{1}{m}.$$

For $y \geq d$, inclusion–exclusion in the inner harmonic sum yields

$$\sum_{\substack{m \leq y \\ (m,d)=1}} \frac{1}{m} = \frac{\varphi(d)}{d} \log y + C_d + O\left(\frac{\tau(d)}{y}\right), \quad C_d \ll (\log(2d))^2.$$

When $T = S$ and $d \leq U$, we have $T^{1/2} > d$ for all sufficiently large X . Apply this formula for $k \leq T^{1/4}$ and estimate the remaining k trivially. Completing the absolutely convergent k -sum gives

$$Q_d(T) = \frac{\log T}{\zeta(2)} \prod_{\ell|d} \left(1 + \frac{1}{\ell}\right)^{-1} + O\left((\log(2d))^2 + T^{-1/4} \log T\right). \quad (30)$$

Indeed, the coefficient of $\log T$ is

$$\frac{\varphi(d)}{d} \sum_{\substack{k \geq 1 \\ (k,d)=1}} \frac{\mu(k)}{k^2} = \frac{1}{\zeta(2)} \prod_{\ell|d} \frac{1 - \ell^{-1}}{1 - \ell^{-2}} = \frac{1}{\zeta(2)} \prod_{\ell|d} \left(1 + \frac{1}{\ell}\right)^{-1}.$$

The product in the main term is at least $\prod_{\ell \leq z} (1 + \ell^{-1})^{-1} \gg 1/\log z$. The error in (30) is therefore

$$o\left(\log T \prod_{\ell|d} (1 + \ell^{-1})^{-1}\right)$$

uniformly when $T = S = X^\eta$ and $d \leq U = (\log X)^8$. Thus

$$\sum_{\substack{s \leq S \\ s \text{ squarefree} \\ (s, d)=1}} \frac{1}{s} \asymp \log S \prod_{\ell|d} \left(1 + \frac{1}{\ell}\right)^{-1}$$

with absolute comparison constants throughout the required range.

Thus the lower bound is

$$\gg \log S \sum_{\substack{d \leq U \\ d|P(z) \\ d \text{ odd}}} \prod_{\ell|d} \frac{1}{\ell+1}.$$

The corresponding unrestricted sum is the Euler product

$$\prod_{3 \leq \ell \leq z} \left(1 + \frac{1}{\ell+1}\right) \asymp \log z.$$

The truncation $d \leq U = z^2$ keeps a fixed positive proportion. Put $w(d) = \prod_{\ell|d} (\ell+1)^{-1}$ and $\alpha = (2 \log z)^{-1}$. Rankin's inequality gives

$$\sum_{\substack{d > U \\ d|P(z) \\ d \text{ odd}}} w(d) \leq U^{-\alpha} \prod_{3 \leq \ell \leq z} \left(1 + \frac{\ell^\alpha}{\ell+1}\right)$$

and hence, after division by the unrestricted product,

$$\frac{\sum_{\substack{d > U \\ d|P(z) \\ d \text{ odd}}} w(d)}{\prod_{3 \leq \ell \leq z} (1 + (\ell+1)^{-1})} \leq e^{-1} \prod_{3 \leq \ell \leq z} \left(1 + \frac{\ell^\alpha - 1}{\ell+2}\right).$$

For $3 \leq \ell \leq z$ we have $0 \leq \alpha \log \ell \leq 1/2$, and $e^t - 1 \leq 1.3t$ on $0 \leq t \leq 1/2$. Therefore the logarithm of the right-hand side is at most

$$-1 + 1.3\alpha \sum_{\ell \leq z} \frac{\log \ell}{\ell+2} = -1 + \frac{1.3}{2} + o(1) < -\frac{1}{3}$$

for all sufficiently large z . The tail is consequently at most $e^{-1/3}$ times the unrestricted product, so the sum over $d \leq U$ is $\gg \log z$. This proves the lower bound. $\square$

Lemma 8.5 (small-prime saturation average). *Uniformly for every $b \in (\mathbb{Z}/P(z)\mathbb{Z})^\times$,*

$$\Sigma_b := \sum_{\substack{d_- \leq U \\ d_- | P(z) \\ d_- \text{ odd}}} \sum_{\substack{e \in \mathcal{E}_{\text{lin}} \\ b \equiv -4e \pmod{d_-}}} \frac{1}{\varphi(4\rho(e))} \asymp (\log X)^2 \log z.$$

Proof. Write $e = rs^2$. For fixed s , observe that a divisor d_- contributes only when $(d_-, s) = 1$, in which case the congruence is the reduced linear progression

$$r \equiv -b(4s^2)^{-1} \pmod{d_-}.$$

Since $\rho(e) = rs$ is squarefree,

$$2\varphi(rs) \leq \varphi(4rs) \leq 4\varphi(rs);$$

the two cases correspond to rs being odd or even. Thus $\varphi(4rs)^{-1} \asymp \varphi(r)^{-1} \varphi(s)^{-1}$ with absolute constants. By [Lemma 8.3](#),

$$\sum_{\substack{r \in (Y_0/s, H/s^2] \\ r \text{ squarefree}, (r, s)=1 \\ b \equiv -4rs^2 \pmod{d_-}}} \frac{1}{\varphi(4rs)} \asymp \frac{1}{sd_-} \log \frac{H}{Y_0s}.$$

Since $\log(H/(Y_0s)) \asymp \log X$ for all $s \leq S$, summing over d_- and using [Lemma 8.4](#) we obtain

$$\Sigma_b \asymp \log X \sum_{\substack{s \leq S \\ s \text{ squarefree}}} \frac{A_s}{s} \asymp \log X \log S \log z \asymp (\log X)^2 \log z.$$

Every comparison has a threshold independent of b , because b enters only through a reduced residue class modulo d_- . $\square$

Proposition 8.6 (saturated cubic event mass). *Uniformly for every $b \in (\mathbb{Z}/P(z)\mathbb{Z})^\times$,*

$$\mu_b \asymp (\log X)^3.$$

Proof. Put

$$\mathcal{P}_X = \sum_{X^\beta < p \leq X^{1-\sigma}} \frac{1}{p}$$

and, for $(a, q) = 1$, define the reciprocal-prime discrepancy

$$\mathcal{E}_X(q, a) = \sum_{\substack{X^\beta < p \leq X^{1-\sigma} \\ p \equiv a \pmod{q}}} \frac{1}{p} - \frac{\mathcal{P}_X}{\varphi(q)}.$$

Mertens' theorem for primes gives

$$\mathcal{P}_X = \log \frac{1-\sigma}{\beta} + o(1). \quad (31)$$

The limiting constant is positive because $\beta < 1 - \sigma$.

For each admissible triple (e, d_-, d_+) , the prime condition (15) is the reduced class

$$p \equiv a(e, d_-, d_+) := -(d_- d_+)^{-1} \pmod{4\rho(e)}.$$

The inverse exists by (14). The coprimality of d_- with $4\rho(e)$ also follows from the small congruence: if an odd prime divided both d_- and e , then $b \equiv -4e \pmod{d_-}$ would contradict $b \in (\mathbb{Z}/P(z)\mathbb{Z})^\times$. The integers e and $\rho(e)$ have the same prime support.

Substitution of the definition of $\mathcal{E}_X$ into (16) therefore gives the exact decomposition

$$\mu_b = \mathcal{P}_X \mathcal{M}_b + \mathcal{R}_b, \quad (32)$$

where

$$\mathcal{M}_b = \sum_{\substack{d_- \leq U \\ d_- | P(z) \\ d_- \text{ odd}}} \sum_{\substack{e \in \mathcal{E}_{\text{in}} \\ b \equiv -4e \pmod{d_-}}} \frac{1}{\varphi(4\rho(e))} \sum_{\substack{d_+ \leq Y/d_- \\ d_+ \text{ squarefree} \\ (d_+, P(z))=1 \\ (d_+, 4\rho(e))=1}} \frac{1}{d_+}, \quad (33)$$

and

$$\mathcal{R}_b = \sum_{\substack{d_- \leq U \\ d_- | P(z) \\ d_- \text{ odd}}} \sum_{\substack{e \in \mathcal{E}_{\text{in}} \\ b \equiv -4e \pmod{d_-}}} \sum_{\substack{d_+ \leq Y/d_- \\ d_+ \text{ squarefree} \\ (d_+, P(z))=1 \\ (d_+, 4\rho(e))=1}} \frac{\mathcal{E}_X(4\rho(e), a(e, d_-, d_+))}{d_+}.$$

We first control $\mathcal{R}_b$. Choose

$$\theta < \nu < \frac{\beta}{2},$$

which is possible by (10). Since $\rho(e) = rs \leq H/s \leq X^\theta$, we have $4\rho(e) \leq X^\nu$ once X is large. For fixed $m = \rho(e)$, the number of possible e is at most $\tau(m)$: the squarefree integer $m = rs$ and the divisor $s \mid m$ determine $e = ms$. There are at most U choices for d_- , while Lemma 7.5, with $M = 4m$, gives

$$\sum_{\substack{d_+ \leq Y/d_- \\ d_+ \text{ squarefree} \\ (d_+, P(z))=1 \\ (d_+, 4m)=1}} \frac{1}{d_+} \ll \frac{\log X}{\log z}.$$

Consequently

$$|\mathcal{R}_b| \ll U \frac{\log X}{\log z} \sum_{\substack{m \\ 4m \leq X^\nu}} \tau(m) \max_{(a, 4m)=1} |\mathcal{E}_X(4m, a)|.$$

Reindexing by $q = 4m$ and using $\tau(m) \leq \tau(q)$ bounds the last sum by

$$\sum_{q \leq X^\nu} \tau(q) \max_{(a, q)=1} |\mathcal{E}_X(q, a)|.$$

Apply [Lemma 7.9](#) with $C = 1$ and $A = 10$. Since $U = (\log X)^8$, we obtain

$$|\mathcal{R}_b| \ll \frac{(\log X)^9}{\log z} (\log X)^{-10} = o((\log X)^3), \quad (34)$$

uniformly in b .

It remains to estimate $\mathcal{M}_b$. Denote its innermost d_+ -sum by $K(e, d_-)$. Extending the range to $d_+ \leq X^\sigma$ and applying [Lemma 7.5](#) with $M = 4\rho(e)$ gives

$$K(e, d_-) \ll \frac{\log X}{\log z}$$

uniformly, since $4\rho(e) \leq 4X^\theta$. For the reverse inequality, restrict to

$$X^{\sigma/2} < d_+ \leq X^{3\sigma/4}.$$

For all sufficiently large X , $d_- \leq U = (\log X)^8 < X^{\sigma/4}$, so this entire interval lies below Y/d_- . Applying (22) with $a = \sigma/2$, $b = 3\sigma/4$, and $M = 4\rho(e)$ gives

$$K(e, d_-) \gg \frac{\log X}{\log z}$$

with the same uniformity. Hence

$$K(e, d_-) \asymp \frac{\log X}{\log z}.$$

By [Lemma 8.5](#),

$$\mathcal{M}_b \asymp \frac{\log X}{\log z} \Sigma_b \asymp \frac{\log X}{\log z} (\log X)^2 \log z \asymp (\log X)^3$$

uniformly in b . Combining this with (31), (32), and (34) proves the proposition. $\square$

Proposition 8.7 (saturated event-level tensorisation). *Let $b \in (\mathbb{Z}/P(z)\mathbb{Z})^\times$. Let $D \leq Y$ be squarefree with $(D, P(z)) = 1$, and let $(c, D) = 1$. Define*

$$B_{D,c}^{(b)} = \sum_{\substack{i=(e,d_-,d_+,p) \in \mathcal{I}_b \\ D|d_+ \\ e \equiv c \pmod{D}}} w_i^{(b)}.$$

Then, uniformly in b, D, c in these ranges, $B_{D,c}^{(b)} \ll \mu_b/D^2$. Moreover,

$$\sum_{\substack{i=(e,d_-,d_+,p) \in \mathcal{I}_b \\ D|d_+}} w_i^{(b)} \ll \frac{\mu_b}{D}.$$

Proof. Choose a fixed ν with $\theta < \nu < \beta$. Since $4\rho(e) \leq X^\nu$ for all sufficiently large X , [Lemma 7.8](#) gives, for fixed e, d_-, d_+ ,

$$\sum_{\substack{X^\beta < p \leq X^{1-\sigma} \\ d_- d_+ p \equiv -1 \pmod{4\rho(e)}}} \frac{1}{d_+ p} \ll \frac{1}{d_+ \varphi(4\rho(e))}.$$

If $D \mid d_+$, write $d_+ = Dt$. The defining coprimality condition removes all terms with $(D, \rho(e)) > 1$. In the remaining terms,

$$\sum_{\substack{d_+ \leq Y/d_- \\ D|d_+ \\ d_+ \text{ squarefree} \\ (d_+, P(z))=1 \\ (d_+, 4\rho(e))=1}} \frac{1}{d_+} \leq \frac{1}{D} \sum_{\substack{t \leq Y \\ t \text{ squarefree} \\ (t, P(z))=1}} \frac{1}{t} \ll \frac{1}{D} \frac{\log X}{\log z}$$

by [Lemma 7.5](#). This is the first factor D^{-1} .

Because $(D, P(z)) = 1$ and $d_- \mid P(z)$, we have $(D, d_-) = 1$. The congruences

$$e \equiv c \pmod{D}, \quad e \equiv -b4^{-1} \pmod{d_-}$$

therefore combine into one class modulo Dd_- . It is reduced: the first residue is reduced by hypothesis, and the second because b is a unit modulo $P(z)$. Also $Dd_- \leq YU$, so [Theorem 8.2](#) applies. Since $\varphi(4\rho(e)) \geq 2\varphi(\rho(e))$, it gives

$$\sum_{\substack{e \in \mathcal{E}_{\text{fin}} \\ e \equiv c \pmod{D} \\ b \equiv -4e \pmod{d_-}}} \frac{1}{\varphi(4\rho(e))} \ll \frac{M_\varphi}{Dd_-}.$$

Consequently

$$\begin{aligned} B_{D,c}^{(b)} &\ll \frac{1}{D} \frac{\log X}{\log z} \sum_{\substack{d_- \leq U \\ d_- | P(z) \\ d_- \text{ odd}}} \frac{M_\varphi}{D d_-} \\ &\ll \frac{M_\varphi \log X}{D^2} \ll \frac{(\log X)^3}{D^2} \asymp \frac{\mu_b}{D^2}, \end{aligned}$$

where $\sum_{d_- \leq U, d_- | P(z)} d_-^{-1} \ll \log z$.

For the second assertion, omit the congruence modulo D and apply [Theorem 8.2](#) only to the reduced class modulo d_- . The same calculation then contains the rough-variable factor D^{-1} but no second factor from the e -sum, and gives

$$\sum_{\substack{i=(e,d_-,d_+,p) \in \mathcal{I}_b \\ D | d_+}} w_i^{(b)} \ll \frac{M_\varphi \log X}{D} \ll \frac{\mu_b}{D}.$$

All constants and thresholds are independent of b, D, c in their stated ranges. □

9 Growing-order Brun coefficients

For the rest of [Sections 9 to 11](#), fix a reduced class $b \pmod{P(z)}$ and suppress it from the notation. Thus $\mathcal{I} = \mathcal{I}_b$, $w_i = w_i^{(b)}$, $\mu = \mu_b$, and $q_i = d_{+,i} p_i$ is the conditional modulus. Two events are compatible if their conditional residue classes agree modulo the gcd of their conditional moduli. Thus $i = (e, d_-, d_+, p)$ and $j = (f, d'_-, d'_+, p')$ are compatible exactly when

$$4e \equiv 4f \pmod{(q_i, q_j)}.$$

Since these conditional moduli are odd, this is equivalent to $e \equiv f \pmod{(q_i, q_j)}$, and hence to the corresponding congruence modulo each prime power dividing the gcd.

Lemma 9.1 (large-prime uniqueness). *Two distinct compatible events in $\mathcal{I}_b$ cannot share the same large prime p .*

Proof. Suppose (e, d_-, d_+, p) and (f, d'_-, d'_+, p) are compatible. Compatibility at p gives $e \equiv f \pmod{p}$. Since $e, f \leq H = X^\theta < X^\beta < p$, we have $e = f$. The defining congruences then give

$$d_- d_+ p \equiv d'_- d'_+ p \equiv -1 \pmod{4\rho(e)}.$$

Because $p > X^\beta > X^\theta \geq \rho(e)$ and p is a large prime, we have $(p, 4\rho(e)) = 1$, so $d_- d_+ \equiv d'_- d'_+ \pmod{4\rho(e)}$. Both total medium factors are at most $Y < Y_0 < \rho(e)$, so they are equal; the small/rough factorization is unique. Thus the two events are identical. □

For compatible $S \subset \mathcal{I}_b$ and $i \notin S$, put

$$L_S = \text{lcm}(q_j : j \in S), \quad g(i; S) = (q_i, L_S).$$

By [Lemma 9.1](#), if $S \cup \{i\}$ is compatible, then $g(i; S) \mid d_{+,i}$.

Lemma 9.2 (sequential appendage). *For every fixed $K > 0$, uniformly in the reduced class b and in compatible $S \subset \mathcal{I}_b$ with $|S| \leq K\mu_b$,*

$$A(S) := \sum_{\substack{i \in \mathcal{I}_b \setminus S \\ S \cup \{i\} \text{ compatible}}} w_i g(i; S) \leq \mu_b (1 + o_K(1)).$$

Proof. Write $i = (e_i, d_{-,i}, d_{+,i}, p_i)$. By [Lemma 9.1](#), if $S \cup \{i\}$ is compatible then the common divisor $g(i; S) = (q_i, L_S)$ has no large-prime factor, hence $g(i; S) \mid d_{+,i}$. For a residual medium prime $\ell > z$, let $C_\ell(S)$ be the number of residue classes modulo ℓ represented by the values e_j with $j \in S$ and $\ell \mid d_{+,j}$. Thus $0 \leq C_\ell(S) \leq \#\{j \in S : \ell \mid d_{+,j}\}$. If D is squarefree, $(D, P(z)) = 1$, and $D \mid g(i; S)$, then for each $\ell \mid D$ compatibility forces $e_i \equiv e_j \pmod{\ell}$ for some such $j \in S$. By the Chinese remainder theorem, e_i is therefore restricted to at most

$$C_D(S) := \prod_{\ell \mid D} C_\ell(S)$$

reduced classes modulo D ; each class is reduced, since $\ell \mid d_{+,j}$ and $(d_{+,j}, 4\rho(e_j)) = 1$ give $\ell \nmid e_j$, e_j and $\rho(e_j)$ having the same prime support. If any $C_\ell(S) = 0$ then there is no contribution.

Use the divisor identity $g = \sum_{D|g} \varphi(D)$. Every divisor that occurs is squarefree, is coprime to $P(z)$, and is at most Y , because $D \mid g(i; S) \mid d_{+,i}$. The $D = 1$ term is at most $\sum_i w_i = \mu$. For $D > 1$, the class count above together with [Proposition 8.7](#) gives

$$\sum_{\substack{i: S \cup \{i\} \text{ compatible} \\ D \mid g(i; S)}} w_i \leq \sum_{c \in C_D(S)} B_{D,c}^{(b)} \ll C_D(S) \frac{\mu}{D^2},$$

where $C_D(S)$ is the set of admissible classes and has size at most $C_D(S)$. Hence the total contribution of this D is $\ll \varphi(D) C_D(S) \mu / D^2 \leq \mu C_D(S) / D$. Therefore

$$A(S) \leq \mu \left(1 + O \left(\sum_{D>1} \frac{C_D(S)}{D} \right) \right).$$

Since the terms are nonnegative, we may enlarge the sum from $D \leq Y$ to all squarefree D whose prime factors exceed z . The Euler product for the squarefree D 's gives

$$1 + \sum_{D>1} \frac{C_D(S)}{D} \leq \prod_{\ell>z} \left(1 + \frac{C_\ell(S)}{\ell} \right) \leq \exp \left(\sum_{\ell>z} \frac{C_\ell(S)}{\ell} \right).$$

Moreover

$$\sum_{\ell>z} \frac{C_\ell(S)}{\ell} \leq \sum_{j \in S} \sum_{\ell \mid d_{+,j}} \frac{1}{\ell} \leq |S| \frac{\log Y}{z \log z}.$$

The last inequality uses $d_{+,j} \leq Y$ and $P^-(d_{+,j}) > z$: such a number has at most $\log Y / \log z$ prime factors, each contributing at most $1/z$. Since $|S| \leq K\mu$, $\mu \asymp (\log X)^3$, $z = (\log X)^4$, and $\log Y \asymp \log X$, the last display is $O_K(1/\log \log X) = o_K(1)$. Thus $A(S) \leq \mu_b(1 + o_K(1))$, uniformly in b and in all compatible $S \subset \mathcal{I}_b$ with $|S| \leq K\mu_b$. $\square$

For $r \geq 0$ define

$$F_r = \sum_{\substack{S \subset \mathcal{I}_b \\ |S|=r \\ S \text{ compatible}}} \frac{1}{\text{lcm}(q_i : i \in S)}, \quad F_0 = 1.$$

Theorem 9.3 (Brun coefficient bound). *For every fixed $K > 0$, uniformly in the reduced class b and for $0 \leq r \leq K\mu_b$,*

$$F_r \leq \frac{\mu_b^r}{r!} \exp\{o_K(\mu_b)\}.$$

Proof. Let

$$\varepsilon_X(K) = \sup_{b \in (\mathbb{Z}/P(z)\mathbb{Z})^\times} \sup_{\substack{S \subset \mathcal{I}_b \text{ compatible} \\ |S| \leq K\mu_b}} \max \left(0, \frac{A(S)}{\mu_b} - 1 \right).$$

By [Lemma 9.2](#), $\varepsilon_X(K) = o_K(1)$. We now derive the recurrence explicitly. If $L_S = \text{lcm}(q_j : j \in S)$ and $S \cup \{i\}$ is compatible, then

$$\frac{1}{\text{lcm}(L_S, q_i)} = \frac{1}{L_S} \frac{(q_i, L_S)}{q_i} = \frac{1}{L_S} w_i g(i; S).$$

Summing this identity over all ways to delete one element from a compatible r -set gives, for $1 \leq r \leq K\mu$,

$$\begin{aligned} r F_r &= \sum_{\substack{S \text{ compatible} \\ |S|=r-1}} \frac{1}{L_S} \sum_{\substack{i \notin S \\ S \cup \{i\} \text{ compatible}}} w_i g(i; S) \\ &\leq \mu(1 + \varepsilon_X(K)) F_{r-1}. \end{aligned}$$

Iterating the recurrence gives

$$F_r \leq \frac{\mu^r}{r!} (1 + \varepsilon_X(K))^r \leq \frac{\mu^r}{r!} \exp\{K\mu\varepsilon_X(K) + O_K(\mu\varepsilon_X(K)^2)\}.$$

Since $\varepsilon_X(K) = o_K(1)$, the exponential factor is $\exp\{o_K(\mu)\}$, uniformly in b and for every $r \leq K\mu_b$. $\square$

10 The base-conditioned CRT no-hit probability

Fix b as above. Let Ω_b be the finite product probability space over all primes appearing in the conditional moduli $q_i = d_{+,i}p_i$. Explicitly,

$$\Omega_b = \prod_{\ell \mid \prod_{i \in I_b} q_i} \mathbb{Z}/\ell\mathbb{Z},$$

with independent uniform coordinates. We regard $\mathcal{A}_i^{(b)}$ as the cylinder event on which the CRT residue is $-4e_i \pmod{q_i}$. Thus

$$\mathbb{P}(\mathcal{A}_i^{(b)}) = \frac{1}{q_i} = w_i^{(b)}.$$

If two events are compatible, the CRT gives

$$\mathbb{P}(\mathcal{A}_i^{(b)} \cap \mathcal{A}_j^{(b)}) = \frac{1}{\text{lcm}(q_i, q_j)} = \frac{(q_i, q_j)}{q_i q_j};$$

if they are incompatible, the intersection is empty. Events with coprime moduli depend on disjoint coordinates, so the graph below is a dependency graph in the sense required by Suen's inequality. Put $W_b = \sum_{i \in I_b} \mathbf{1}_{\mathcal{A}_i^{(b)}}$. For distinct events $i \neq j$, let $i \sim j$ denote $(q_i, q_j) > 1$, and define

$$\delta_b = \sup_i \sum_{\substack{j \sim i \\ j \neq i}} \mathbb{P}(\mathcal{A}_j^{(b)}), \quad \Delta_b = \sum_{\substack{\{i,j\}: i \sim j \\ i \neq j}} \mathbb{P}(\mathcal{A}_i^{(b)} \cap \mathcal{A}_j^{(b)}).$$

Proposition 10.1 (mass-dependency compatibility scale). *Uniformly in the base class b , the saturated mass law is compatible with the rough residual dependency graph in the precise sense that*

$$\mu_b \frac{\log Y}{z \log z} = o(1), \quad \mu_b \sum_{\substack{D > 1 \\ D \text{ squarefree} \\ (D, P(z))=1}} \frac{1}{D^2} = o(1).$$

Consequently

$$\mu_b^2 \sum_{\substack{D > 1 \\ D \text{ squarefree} \\ (D, P(z))=1}} \frac{1}{D^2} = o(\mu_b).$$

Proof. By Proposition 8.6, $\mu_b \asymp (\log X)^3$. Also $Y = X^\sigma$, so $\log Y \asymp \log X$, while $z = (\log X)^4$ and $\log z \asymp \log \log X$. Hence

$$\mu_b \frac{\log Y}{z \log z} \ll (\log X)^3 \frac{\log X}{(\log X)^4 \log \log X} \ll \frac{1}{\log \log X} = o(1).$$

For the second assertion, every squarefree $D > 1$ with $(D, P(z)) = 1$ has a prime factor $p > z$. Therefore

$$\sum_{\substack{D > 1 \\ D \text{ squarefree} \\ (D, P(z))=1}} \frac{1}{D^2} \leq \prod_{p > z} \left(1 + \frac{1}{p^2}\right) \sum_{p > z} \frac{1}{p^2} \ll \frac{1}{z \log z},$$

by partial summation and the prime number theorem. Multiplying by $\mu_b \asymp (\log X)^3$ gives $\mu_b/(z \log z) \ll 1/((\log X) \log \log X) = o(1)$, and the final display follows by multiplying once more by μ_b . $\square$

Lemma 10.2 (large-prime-neighbour bound). *Fix ε with $0 < \varepsilon < \beta/2$ and a prime $p > X^\beta$. The total weight of the events $j = (f, \delta_-, \delta_+, p) \in I_b$ that share the large prime p satisfies*

$$\sum_{\substack{j \in I_b \\ j \text{ has large prime } p}} w_j \ll X^\varepsilon p^{-1} (\log X)^{O(1)} = o(1),$$

uniformly in b and in p .

Proof. Since $Y < X^\beta < p$, every event that shares a large prime with a fixed event has that same large prime p . For $j = (f, \delta_-, \delta_+, p)$ the defining congruence reads $4\rho(f) \mid \delta_- \delta_+ p + 1$, and $w_j = 1/(\delta_+ p)$, so the total weight is at most

$$\frac{1}{p} \sum_{\substack{\delta_- \leq U, \delta_- \mid P(z) \\ \delta_- \text{ squarefree and odd } \delta_+ \text{ squarefree, } (\delta_+, P(z))=1}} \sum_{\delta_+ \leq Y/\delta_-} \frac{1}{\delta_+} \# \{f \in \mathcal{E}_{\text{lin}} : 4\rho(f) \mid \delta_- \delta_+ p + 1\}. \quad (35)$$

Count the inner set first. Fix the total medium factor $d' = \delta_- \delta_+ \leq Y$. If the inner set is nonempty, then $4\rho(f) \mid d'p + 1$, so $M = (d'p + 1)/4$ is a positive integer. If $\rho(f) = m$, then $m \mid M$; once $m = rs$ is fixed, the exact-divisor shape $f = rs^2 = ms$ leaves at most $\tau(m)$ choices for s . Hence

$$\#\{f \in \mathcal{E}_{\text{lin}} : 4\rho(f) \mid d'p + 1\} \leq \sum_{m \mid M} \tau(m) \leq \tau(M)^2 \ll_{\varepsilon} X^{\varepsilon},$$

since $d'p \leq YX^{1-\sigma} \leq X$. The remaining sum over the medium factor is

$$\sum_{\delta_- \leq U, \delta_- \mid P(z)} \sum_{\delta_+ \leq Y/\delta_-, (\delta_+, P(z))=1} \frac{1}{\delta_+} \ll U \frac{\log X}{\log z} = (\log X)^{O(1)}.$$

Feeding the two bounds into (35) gives $\ll_{\varepsilon} X^{\varepsilon} p^{-1} (\log X)^{O(1)} = o(1)$, since $p > X^{\beta}$ and $\varepsilon < \beta/2$. $\square$

Lemma 10.3 (dependency bounds). *Uniformly in b , $\delta_b = o(1)$ and $\Delta_b = o(\mu_b)$.*

Proof. Apply the second assertion of Proposition 8.7 with $D = \ell$ for each prime $\ell \mid d_+$. The medium-prime neighbours of $i = (e, d_-, d_+, p)$ contribute

$$\sum_{\ell \mid d_+} \sum_{\substack{j \in \mathcal{I}_b \\ \ell \mid d_{+,j}}} w_j \ll \mu_b \sum_{\ell \mid d_+} \frac{1}{\ell} \ll \mu_b \frac{\log Y}{z \log z} = o(1),$$

by Proposition 10.1. The large-prime neighbours of i fall under Lemma 10.2, applied with $\varepsilon = \beta/4$: for the fixed large prime $p > X^{\beta}$ of i , the total weight of the events sharing it, the self-event included, is $o(1)$ uniformly in b .

Turn to Δ_b . A distinct compatible dependent pair cannot share a large prime, so it must share a nontrivial residual medium divisor. Expanding over that divisor,

$$\Delta_b \ll \sum_{\substack{D > 1 \\ D \leq Y \\ D \text{ squarefree} \\ (D, P(z))=1}} \varphi(D) \sum_{\substack{c \pmod{D} \\ (c, D)=1}} (B_{D,c}^{(b)})^2.$$

By Proposition 8.7,

$$\max_{\substack{c \pmod{D} \\ (c, D)=1}} B_{D,c}^{(b)} \ll \frac{\mu_b}{D^2}, \quad \sum_{\substack{c \pmod{D} \\ (c, D)=1}} B_{D,c}^{(b)} \ll \frac{\mu_b}{D}.$$

Therefore

$$\Delta_b \ll \mu_b^2 \sum_{\substack{D > 1 \\ D \text{ squarefree} \\ (D, P(z))=1}} \frac{1}{D^2} \ll \frac{\mu_b^2}{z \log z} = o(\mu_b),$$

since $z = (\log X)^4$ and $\mu_b \asymp (\log X)^3$. $\square$

Theorem 10.4 (base-conditioned CRT no-hit estimate). *Uniformly in b ,*

$$\mathbb{P}(W_b = 0) \leq \exp\{-(1 - o(1))\mu_b\}.$$

Proof. Suen's correlation inequality [14, 10], with the unordered pair convention for Δ_b , gives

$$\mathbb{P}(W_b = 0) \leq \exp\{-\mu_b + O(\Delta_b e^{2\delta_b})\}.$$

Versions stated with ordered pairs differ by only an absolute factor in Δ_b , which is absorbed in the $O(\cdot)$ term. Use Lemma 10.3. $\square$

11 Transfer to finite intervals

Fix $b \in (\mathbb{Z}/P(z)\mathbb{Z})^{\times}$. For $r \geq 0$ define

$$C_r^{(b)}(N) = \sum_{\substack{S \subset \mathcal{I}_b \\ |S|=r}} \#\{n \leq N : n \equiv b \pmod{P(z)}, n \in \mathcal{A}_i^{(b)} \text{ for all } i \in S\}.$$

Lemma 11.1 (finite-interval approximation in one base class). *For every $r \geq 0$,*

$$C_r^{(b)}(N) = \frac{N}{P(z)} F_r + O(X^r F_r).$$

Proof. If S is compatible, the residual intersection is one class modulo $L_S = \text{lcm}(q_i : i \in S)$. Since $(L_S, P(z)) = 1$, the base condition gives one class modulo $P(z)L_S$, contributing $N/(P(z)L_S) + O(1)$. Incompatible intersections are empty. The number of compatible sets is at most $X^r F_r$. $\square$

Theorem 11.2 (saturated finite-interval transfer). *Let $K > 0$ be a fixed sufficiently large constant and put $R_b = \lfloor K\mu_b \rfloor$. Suppose that*

$$P(z)X^{2R_b} \exp\{2\mu_b\} \leq N \exp\{-4\mu_b\}. \quad (36)$$

Then, uniformly in b ,

$$\#\{n \leq N : n \equiv b \pmod{P(z)}, n \notin \mathcal{A}_i^{(b)} \text{ for every } i \in \mathcal{I}_b\} \leq \frac{N}{P(z)} \exp\{-(1 - o(1))\mu_b\}.$$

Proof. Write $M_b(N)$ for the left-hand side. The even Bonferroni inequality gives

$$M_b(N) \leq \sum_{r=0}^{2R_b} (-1)^r C_r^{(b)}(N).$$

At a single point this is the elementary inequality $\mathbf{1}_{W=0} \leq \sum_{r \leq 2R} (-1)^r \binom{W}{r}$. We will also need the matching upper bound

$$\sum_{r \leq 2R} (-1)^r \binom{W}{r} \leq \mathbf{1}_{W=0} + \binom{W}{2R}.$$

At a fixed point the number of r -subsets of hit events is $\binom{W}{r}$, so the double count exchanges the sum over points for the sum over r -subsets of events. By Lemma 11.1 the truncated alternating sum splits into its model expectation and the accumulated error:

$$\sum_{r=0}^{2R_b} (-1)^r C_r^{(b)}(N) = \frac{N}{P(z)} \mathbb{E} \left[\sum_{r=0}^{2R_b} (-1)^r \binom{W_b}{r} \right] + O \left(\sum_{r \leq 2R_b} X^r F_r \right).$$

Bound the error by Theorem 9.3 with the fixed constant $2K$:

$$\sum_{r \leq 2R_b} X^r F_r \leq X^{2R_b} \exp\{(1 + o_K(1))\mu_b\} \leq X^{2R_b} e^{2\mu_b},$$

where the $o_K(1)$ is uniform in b . The level hypothesis keeps this below $(N/P(z))e^{-4\mu_b}$. We control the expectation through the companion bound: in the CRT model,

$$\sum_{r=0}^{2R_b} (-1)^r \binom{W_b}{r} \leq \mathbf{1}_{W_b=0} + \binom{W_b}{2R_b}.$$

The first term, after taking expectations, is the no-hit probability estimated in Theorem 10.4. For the second, Theorem 9.3 with the fixed constant $2K$ supplies

$$F_{2R_b} \leq \frac{\mu_b^{2R_b}}{(2R_b)!} \exp\{o_K(\mu_b)\}.$$

With $R_b = K\mu_b + O(1)$, Stirling's formula evaluates the factorial ratio:

$$\log \frac{\mu_b^{2R_b}}{(2R_b)!} = 2K\mu_b(1 - \log(2K)) + O_K(\log \mu_b).$$

Take K large enough that the right-hand side drops below $-4\mu_b$ for all large X ; the surviving factor $\exp\{o_K(\mu_b)\}$ then yields $F_{2R_b} \leq e^{-3\mu_b}$, well inside the target. This proves the theorem. $\square$

12 Optimization, saturation lifting, and proof of the main theorem

Proof of Theorem 1.1. The saturated mass law is $\mu_b \asymp (\log X)^3$, uniformly in b . Thus there are constants $c_\mu, C_\mu > 0$ and X_0 such that

$$c_\mu(\log X)^3 \leq \mu_b \leq C_\mu(\log X)^3$$

for every reduced $b \pmod{P(z)}$ and every $X \geq X_0$. Choose

$$\log X = \alpha(\log N)^{1/4}, \quad (37)$$

where $\alpha > 0$ is a sufficiently small fixed constant. Then

$$\mu_b \asymp_\alpha (\log N)^{3/4}. \quad (38)$$

Since $z = (\log X)^4$, Mertens' theorem gives

$$\log P(z) = \sum_{\ell \leq z} \log \ell \leq 2z \ll \alpha^4 \log N.$$

Also, with $R_b = \lfloor K\mu_b \rfloor$,

$$2R_b \log X + 6\mu_b \leq C_K \mu_b \log X \leq C'_K \alpha^4 \log N.$$

Choosing α sufficiently small makes (36) valid for every reduced $b \pmod{P(z)}$. The constants in the preceding inequalities do not depend on b , and the lower mass bound supplies a single constant $c_1 > 0$ in the exponential saving. Summing Theorem 11.2 over all reduced b gives

$$E_{\text{red}}(N; z) := \#\{n \leq N : (n, P(z)) = 1, n \text{ is missed by all saturated certificates}\} \quad (39)$$

$$\ll N \exp\{-c_1(\log N)^{3/4}\}. \quad (40)$$

We now lift from reduced integers to all integers. Let $\Psi(x, y)$ be the number of y -smooth integers at most x . Rankin's trick with exponent $1/2$ [15] gives

$$\Psi(x, y) \leq x^{1/2} \prod_{p \leq y} (1 - p^{-1/2})^{-1} \ll x^{1/2} \exp\{C\sqrt{y}\}. \quad (41)$$

For each $n \leq N$, write $n = sm$, where every prime factor of s is at most $z_N = (\alpha(\log N)^{1/4})^4$ and $(m, P(z_N)) = 1$. If m has a representation, then n has one after multiplying all denominators by s .

The contribution with $m \leq N^{1/2}$ is

$$\sum_{m \leq N^{1/2}} \Psi(N/m, z_N) \ll e^{C\sqrt{z_N}} N^{1/2} \sum_{m \leq N^{1/2}} m^{-1/2} \ll N^{3/4+o(1)}.$$

If $n = sm$ is exceptional, then m is exceptional: a representation of $4/m$ would become one of $4/n$ after multiplying all three denominators by s . Thus m is missed by the selected family at every scale at which that family is defined. For $m > N^{1/2}$, apply (39) at the scale $M = N/s$ with $z_M = (\alpha(\log M)^{1/4})^4$. Since $M > N^{1/2}$, the saving is still $\exp\{-c_2(\log N)^{3/4}\}$ after changing c_2 . Indeed, $\exp\{-c_1(\log M)^{3/4}\} \leq \exp\{-c_2(\log N)^{3/4}\}$ after adjusting c_2 . Moreover, all such M exceed the same absolute threshold once N is large. Since $z_M \leq z_N$ the condition $(m, P(z_N)) = 1$ implies $(m, P(z_M)) = 1$. Thus the contribution is at most

$$N e^{-c_2(\log N)^{3/4}} \sum_{\substack{s \geq 1 \\ P^+(s) \leq z_N}} \frac{1}{s} \ll N e^{-c_2(\log N)^{3/4}} \prod_{p \leq z_N} \left(1 - \frac{1}{p}\right)^{-1},$$

and the Euler product is $\ll \log z_N$, which is absorbed into the exponential saving. Combining the two ranges gives

$$E(N) \ll N \exp\{-c(\log N)^{3/4}\}.$$

Every saturated hit is a valid divisor-fan certificate by Section 5, so this proves Theorem 1.1. Enlarging the implicit constant absorbs the remaining finite range of N . $\square$

Remark 12.1 (why the logarithmic denominator disappears). *The extra $\log z$ factor over the one-rough-coordinate family comes from averaging $1/d_-$ over exact divisors through the small-prime congruence $b \equiv -4e \pmod{d_-}$; the dependency estimates still see only primes of d_+ , all exceeding $z = (\log X)^4$.*

13 Applications of the saturated fan

Unconditional consequences

Let

$$\mathcal{P}_{\text{exc}}(N) = \{p \leq N : p \text{ is prime and } 4/p \text{ has no three-unit representation}\}.$$

Corollary 13.1 (prime denominators). *There is an absolute constant $c > 0$ such that, for all $N \geq 3$,*

$$\#\mathcal{P}_{\text{exc}}(N) \ll N \exp\{-c(\log N)^{3/4}\}.$$

Consequently

$$\#\mathcal{P}_{\text{exc}}(N) \ll \pi(N) \exp\{-c'(\log N)^{3/4}\}$$

for some $c' > 0$.

Proof. The first bound follows from [Theorem 1.1](#), since prime denominators are a subset of all denominators; the second then uses $\pi(N) \gg N/\log N$, after decreasing the constant in the exponential. $\square$

The next corollary gives a quantitative form of the elementary lifting principle: if a divisor of n is representable, then n is representable after multiplying all three denominators by the complementary factor.

Proposition 13.2 (large-prime-factor propagation). *Let*

$$E_{>Y}(N) = \#\{n \leq N : n \text{ is exceptional for } 4/n, P^+(n) > Y\}.$$

Uniformly for $N \geq 3$ *and* $3 \leq Y \leq N$,

$$E_{>Y}(N) \ll N(1 + \log N) \exp\{-c(\log Y)^{3/4}\}.$$

In particular, for every fixed $\eta > 0$,

$$\#\{n \leq N : n \text{ exceptional}, P^+(n) > N^\eta\} \ll_\eta N \exp\{-c_\eta(\log N)^{3/4}\},$$

and for $Y = \exp\{(\log N)^{3/4}\}$ *one obtains*

$$E_{>Y}(N) \ll N \exp\{-c'(\log N)^{9/16}\}.$$

Proof. If $p \mid n$ and $4/p = 1/x + 1/y + 1/z$, then

$$\frac{4}{n} = \frac{1}{(n/p)x} + \frac{1}{(n/p)y} + \frac{1}{(n/p)z}.$$

Thus every prime divisor of an exceptional integer is itself an exceptional prime. Let $A(t) = \#\mathcal{P}_{\text{exc}}(t)$. By [Corollary 13.1](#), $A(t) \ll t \exp\{-c(\log t)^{3/4}\}$. Hence

$$E_{>Y}(N) \leq N \sum_{\substack{Y < p \leq N \\ p \text{ exceptional}}} \frac{1}{p}.$$

Partial summation gives

$$\sum_{\substack{Y < p \leq N \\ p \text{ exceptional}}} \frac{1}{p} \leq \frac{A(N)}{N} + \int_Y^N \frac{A(t)}{t^2} dt \ll (1 + \log N) \exp\{-c(\log Y)^{3/4}\}.$$

The two displayed specializations follow by absorbing powers of $\log N$ into the exponential saving. $\square$

Fixed numerators: unconditional

For an integer $m \geq 2$, let $E_m(N)$ denote the number of positive integers $n \leq N$ for which

$$\frac{m}{n} \neq \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$$

for all positive integers x, y, z .

Proposition 13.3 (fixed-numerator transfer estimates). *Fix an integer* $m \geq 2$. *For all sufficiently large* X , *with the threshold allowed to depend on* m , *form the saturated family used in* [Section 5](#) *after imposing*

$$(\rho(e), m) = 1, \quad (d_- d_+, m) = 1,$$

and replacing the congruence conditions by

$$d_- d_+ p \equiv -1 \pmod{m\rho(e)}, \quad b \equiv -me \pmod{d_-}.$$

For each resulting quadruple $i = (e, d_-, d_+, p)$ *put*

$$Q_i = d_- d_+ p, \quad q_i = d_+ p, \quad a_i = \frac{Q_i + 1}{m},$$

and replace the residual event by

$$\mathcal{A}_i^{(m,b)} = \{n : n \equiv -me \pmod{q_i}\}, \quad w_i^{(m,b)} = \frac{1}{q_i}.$$

Let $\mu_{m,b}$, $B_{D,c}^{(m,b)}$, $\delta_{m,b}$, and $\Delta_{m,b}$ denote the fixed- m analogues of the mass, tensorisation sum, maximum neighbour mass, and pair-dependency sum from [Propositions 8.6 and 8.7](#) and [Lemma 10.3](#). Then, uniformly in every reduced base class $b \pmod{P(z)}$,

$$\mu_{m,b} \asymp_m (\log X)^3,$$

and for every squarefree $D \leq Y$ with $(D, P(z)m) = 1$ and every reduced class $c \pmod{D}$,

$$B_{D,c}^{(m,b)} \ll_m \frac{\mu_{m,b}}{D^2}, \quad \sum_{\substack{i=(e,d_+,d_+,p) \\ D|d_+}} w_i^{(m,b)} \ll_m \frac{\mu_{m,b}}{D}.$$

Moreover the sequential appendage bound of [Lemma 9.2](#), the Suen dependency bounds $\delta_{m,b} = o_m(1)$ and $\Delta_{m,b} = o_m(\mu_{m,b})$, and the finite-interval transfer of [Theorem 11.2](#) all hold with constants depending on m .

Proof. Fix m and take X large enough that every prime divisor of m is below z . Let $m_0 = \prod_{\ell|m} \ell$. For squarefree s with $(s, m) = 1$, [\(17\)](#), applied with the squarefree modulus sm_0 , gives

$$\sum_{\substack{Y_0/s < r \leq H/s^2 \\ r \text{ squarefree} \\ (r, sm) = 1}} \frac{1}{r} \asymp_m \frac{\varphi(s)}{s} \log \frac{H}{Y_0 s}.$$

The expansion

$$\frac{1}{\varphi(r)} = \frac{1}{r} \sum_{k|r} \mu^2(k) \prod_{\ell|k} \frac{1}{\ell-1}$$

used in [Proposition 8.1](#) is unchanged after imposing $(r, m) = 1$: inclusion–exclusion over the finitely many divisors of m_0 changes only the constants. Summing first in r and then in the squarefree variable s , now restricted by $(s, m) = 1$, therefore gives

$$\sum_{\substack{e \in \mathcal{E}_{\text{lin}} \\ (\rho(e), m) = 1}} \frac{1}{\rho(e)} \asymp_m (\log X)^2, \quad \sum_{\substack{e \in \mathcal{E}_{\text{lin}} \\ (\rho(e), m) = 1}} \frac{1}{\varphi(\rho(e))} \asymp_m (\log X)^2.$$

Thus deleting the primes dividing m preserves both sides of the quadratic exact-divisor estimate.

The small-prime average also retains its order. For fixed admissible s and d_- , the congruence is

$$r \equiv -b(ms^2)^{-1} \pmod{d_-}.$$

It is a reduced class because $(d_-, ms) = 1$. The proof of [Lemma 8.3](#), with the finite condition $(r, m) = 1$ inserted by inclusion–exclusion, has the same main term multiplied by

$$\prod_{\ell|m} \left(1 - \frac{1}{\ell}\right) = \frac{\varphi(m_0)}{m_0} > 0.$$

Since $(m, rs) = 1$, one also has $\varphi(mrs) = \varphi(m)\varphi(r)\varphi(s)$. It follows that

$$\sum_{\substack{Y_0/s < r \leq H/s^2 \\ r \text{ squarefree} \\ (r, sm) = 1 \\ b \equiv -mr s^2 \pmod{d_-}}} \frac{1}{\varphi(mrs)} \asymp_m \frac{1}{sd_-} \log \frac{H}{Y_0 s}.$$

Removing from s and d_- the finitely many primes dividing m changes the Euler products in [Lemma 8.4](#) by positive constants. Consequently the fixed- m analog of the full small-prime sum is

$$\Sigma_{m,b} \asymp_m (\log X)^2 \log z$$

uniformly in the reduced base class b .

We next sum over p . Choose ν with $\theta < \nu < \beta/2$. Since m is fixed and $\rho(e) \leq X^\theta$, eventually $m\rho(e) \leq X^\nu$. The discrepancy sum over the progressions

$$d_- d_+ p \equiv -1 \pmod{m\rho(e)}$$

is bounded by [Lemma 7.9](#). To make the reindexing explicit, let $\mathcal{E}_X(q, a)$ be the reciprocal-prime discrepancy from the proof of [Proposition 8.6](#). For a fixed value $t = \rho(e)$, there are at most $\tau(t)$ possible choices of $e = rs^2$, at most U choices of d_- , and the reciprocal sum over d_+ is $O_m(\log X/\log z)$. Hence the total discrepancy is

$$\ll_m U \frac{\log X}{\log z} \sum_{\substack{q \leq X^\nu \\ m|q}} \tau(q) \max_{(a,q)=1} |\mathcal{E}_X(q, a)| \ll_m \frac{(\log X)^9}{\log z} (\log X)^{-10} = o_m((\log X)^3),$$

where the last estimate is [Lemma 7.9](#) with $C = 1$ and $A = 10$. The main term splits as

$$\frac{1}{\varphi(m\rho(e))} = \frac{1}{\varphi(m)\varphi(\rho(e))}$$

because $(m, \rho(e)) = 1$. In the rough d_+ -sum, [Lemma 7.5](#) and (22) apply with coprimality modulus $m\rho(e)$; the fixed primes of m are already below z , while $(d_+, P(z)) = 1$. Hence this sum is still $\asymp_m \log X/\log z$. Combining the prime-window mass, this rough factor, and the estimate for $\Sigma_{m,b}$ gives

$$\mu_{m,b} \asymp_m (\log X)^3$$

uniformly in b .

For the progression estimates, let $D \mid d_+$ and write $d_+ = Dt$. Brun–Titchmarsh in the modulus $m\rho(e)$ gives

$$\sum_{\substack{X^\beta < p \leq X^{1-\sigma} \\ d_- d_+ p \equiv -1 \pmod{m\rho(e)}}} \frac{1}{d_+ p} \ll_m \frac{1}{d_+ \varphi(\rho(e))}.$$

The rough sum over d_+ contributes D^{-1} . The congruences

$$e \equiv c \pmod{D}, \quad e \equiv -bm^{-1} \pmod{d_-}$$

combine into one reduced class modulo Dd_- , since $(D, d_-m) = 1$. Restricting further to $(\rho(e), m) = 1$ can only decrease the upper sum. Applying [Theorem 8.2](#) therefore contributes a second D^{-1} . More explicitly, put

$$M_{\varphi,m} := \sum_{\substack{e \in \mathcal{E}_{\text{lin}} \\ (\rho(e), m) = 1}} \frac{1}{\varphi(\rho(e))} \asymp_m (\log X)^2.$$

Together with [Proposition 8.1](#), this gives $M_\varphi \ll_m M_{\varphi,m}$. Thus the bound supplied by [Theorem 8.2](#) may be written in terms of $M_{\varphi,m}$:

$$B_{D,c}^{(m,b)} \ll_m \frac{1}{D} \frac{\log X}{\log z} \sum_{\substack{d_- \leq U \\ d_- \mid P(z)}} \frac{M_{\varphi,m}}{Dd_-} \ll_m \frac{M_{\varphi,m} \log X}{D^2},$$

and $\mu_{m,b} \asymp_m (\log X)^3$ gives

$$B_{D,c}^{(m,b)} \ll_m \frac{\mu_{m,b}}{D^2}.$$

Omitting the congruence modulo D removes that second factor; the same calculation gives

$$\sum_{\substack{i=(e,d_-,d_+,p) \\ D \mid d_+}} w_i^{(m,b)} \ll_m \frac{\mu_{m,b}}{D}.$$

It remains to check that the overlap argument survives the multiplier m . Suppose two compatible events share their large prime p . Compatibility gives $me \equiv mf \pmod{p}$. For large X we have $p > m$ and $e, f \leq H < p$, so $e = f$. Cancelling p from the two defining congruences modulo $m\rho(e)$ gives

$$d_- d_+ \equiv d'_- d'_+ \pmod{m\rho(e)}.$$

Both medium factors are at most $Y < \rho(e)$, hence they are equal; their small-prime and rough-prime parts are then equal as well. Thus the large-prime uniqueness lemma remains valid.

There is also a uniform fixed- m version of the large-prime-neighbour bound. For a neighbouring event with medium factor d' , the defining congruence implies

$$\rho(f) \mid \frac{d'p + 1}{m}.$$

The quotient is an integer of size $O_m(X)$. Summing over its divisors gives at most $\tau((d'p+1)/m)^2 \ll_{m,\varepsilon} X^\varepsilon$ possible f , exactly as in [Lemma 10.2](#). The same choice $0 < \varepsilon < \beta/2$ makes the total weight of these neighbours $o_m(1)$.

For every residual prime $\ell \mid d_+$, one has $\ell \nmid m$. Compatibility modulo ℓ is therefore equivalent to $e_i \equiv e_j \pmod{\ell}$, since the residual classes are $-me_i$ and $-me_j$. For a compatible set S , let $C_\ell(S)$ be the number of residue classes modulo ℓ represented by those e_j for which $\ell \mid d_{+,j}$, and put $C_D(S) = \prod_{\ell \mid D} C_\ell(S)$. Writing $A_m(S)$ for the fixed- m appendage sum

$$A_m(S) = \sum_{\substack{i \notin S \\ S \cup \{i\} \text{ compatible}}} w_i^{(m,b)}(q_i, \text{lcm}(q_j : j \in S)),$$

the two progression bounds give

$$A_m(S) \leq \mu_{m,b} \left(1 + O_m \left(\sum_{D>1} \frac{C_D(S)}{D} \right) \right).$$

The Euler-product exponent that bounds the sum is

$$|S| \frac{\log Y}{z \log z} \ll_{K,m} (\log X)^3 \frac{\log X}{(\log X)^4 \log \log X} = o_{K,m}(1),$$

so $A_m(S) \leq \mu_{m,b}(1 + o_{K,m}(1))$. The same progression bounds and the preceding large-prime-neighbour estimate give, uniformly in b ,

$$\delta_{m,b} \ll_m \mu_{m,b} \frac{\log Y}{z \log z} + o_m(1) = o_m(1)$$

and

$$\Delta_{m,b} \ll_m \mu_{m,b}^2 \sum_{\substack{D>1 \\ D \text{ squarefree} \\ (D, P(z))=1}} \frac{1}{D^2} \ll_m \frac{\mu_{m,b}^2}{z \log z} = o_m(\mu_{m,b}).$$

Let $W_{m,b}$ count the fixed- m events in the corresponding CRT probability space. Suen's inequality therefore gives

$$\mathbb{P}(W_{m,b} = 0) \leq \exp\{-(1 - o_m(1))\mu_{m,b}\}.$$

Finally, each $q_i = d_+p$ is squarefree, coprime to $P(z)$, and at most X . Compatible intersections still occupy one CRT row modulo the least common multiple of their residual moduli. Let $F_r^{(m,b)}$ be the sum of the reciprocal least common multiples over compatible r -sets. The appendage bound and the recurrence in [Theorem 9.3](#), invoked with the fixed constant $2K$, give

$$F_r^{(m,b)} \leq \frac{\mu_{m,b}^r}{r!} \exp\{o_{K,m}(\mu_{m,b})\} \quad (r \leq 2K\mu_{m,b}).$$

Define

$$C_r^{(m,b)}(N) = \sum_{\substack{S \subset \mathcal{I}_{m,b} \\ |S|=r}} \#\{n \leq N : n \equiv b \pmod{P(z)}, n \in \mathcal{A}_i^{(m,b)} \text{ for all } i \in S\},$$

where $\mathcal{I}_{m,b}$ is the fixed- m event family. Then

$$C_r^{(m,b)}(N) = \frac{N}{P(z)} F_r^{(m,b)} + O(X^r F_r^{(m,b)}),$$

because each compatible intersection and the base condition determine one class modulo $P(z) \text{lcm}(q_i : i \in S)$. Thus, with $R_{m,b} = \lfloor K\mu_{m,b} \rfloor$, the even Bonferroni truncation in [Theorem 11.2](#) gives

$$\#\{n \leq N : n \equiv b \pmod{P(z)}, n \notin \mathcal{A}_i^{(m,b)} \text{ for all } i\} \leq \frac{N}{P(z)} \exp\{-(1 - o_m(1))\mu_{m,b}\}$$

whenever

$$P(z)X^{2R_{m,b}} \exp\{2\mu_{m,b}\} \leq N \exp\{-4\mu_{m,b}\}.$$

This is the asserted fixed- m finite transfer and completes the proof. $\square$

Theorem 13.4 (fixed-numerator Erdős–Straus–Schinzel exceptional sets). *For each fixed integer $m \geq 2$, there are constants $c_m, C_m > 0$ such that, for all $N \geq 3$,*

$$E_m(N) \leq C_m N \exp\{-c_m(\log N)^{3/4}\}.$$

Proof. The certificate calculation used for [Theorem 1.1](#) has a fixed-numerator variant. If $a \geq 1$, $e \geq 1$, and

$$q = ma - 1, \quad e \mid a^2, \quad n \equiv -me \pmod{q},$$

and

$$x = an, \quad y = \frac{an + e}{q}, \quad z = \frac{an + a^2 n^2 / e}{q}.$$

then $ma \equiv 1 \pmod{q}$, so $an \equiv -e \pmod{q}$. Also $(e, q) = 1$, because every prime dividing e divides a and hence cannot divide $q = ma - 1$. The same calculation as in [Lemma 2.3](#) gives

$$\frac{1}{y} + \frac{1}{z} = \frac{q}{an}, \quad \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1+q}{an} = \frac{m}{n}.$$

In each event from the fixed- m saturated family put $Q = d_- d_+ p$. The congruence $Q \equiv -1 \pmod{m\rho(e)}$ makes $a = (Q + 1)/m$ an integer with $\rho(e) \mid a$; since $e = rs^2$ and $\rho(e) = rs$, this implies $e \mid a^2$. The conditioned base class and the residual class, together with the coprimality $(d_-, d_+ p) = 1$ supplied by the saturated-family constraints, combine to give $n \equiv -me \pmod{Q}$, so the certificate above applies with $q = Q$. Every reduced base class lies in the solubility range of [Proposition 13.3](#): taking $d_- = 1$ makes the small congruence $b \equiv -me \pmod{d_-}$ vacuous, while the proposition supplies the full fixed- m mass, tensorisation, appendage, dependency, and finite-interval estimates in those classes.

We now carry out the lift. The proposition and the finite-transfer argument give constants $c_{m,0}, C_{m,0} > 0$ such that, for all sufficiently large M ,

$$\#\{u \leq M : (u, P(z_M)) = 1, u \text{ is missed}\} \leq C_{m,0} M \exp\{-c_{m,0}(\log M)^{3/4}\},$$

where

$$\log X_M = \alpha_m (\log M)^{1/4}, \quad z_M = (\log X_M)^4,$$

and $\alpha_m > 0$ is chosen small enough for the level condition. Factor each $n \leq N$ uniquely as $n = su$, where all prime factors of s are at most z_N and $(u, P(z_N)) = 1$. If m/u has a three-unit representation, then multiplying its denominators by s gives one for m/n ; hence an exceptional n forces u to be exceptional and therefore missed.

For $u \leq N^{1/2}$, Rankin's estimate (41) gives a total of $N^{3/4+o_m(1)}$ possible n . For $u > N^{1/2}$, put $M = N/s$. Then $M > N^{1/2}$, $z_M \leq z_N$, and $(u, P(z_N)) = 1$ implies $(u, P(z_M)) = 1$. Applying the displayed reduced estimate at scale M and summing over z_N -smooth s gives

$$\ll_m N \exp\{-c_{m,1}(\log N)^{3/4}\} \sum_{\substack{s \geq 1 \\ P^+(s) \leq z_N}} \frac{1}{s} \ll_m N \exp\{-c_{m,2}(\log N)^{3/4}\},$$

because the Euler product over $p \leq z_N$ is $O(\log z_N)$. The first range is smaller than the same bound for large N . Enlarging C_m covers the remaining finite range. Every constant and threshold may depend on the fixed numerator m ; no uniformity in a growing numerator is claimed. $\square$

Growing numerators: conditional uniformity

The results of this subsection and the next are conditional on additional hypotheses and are independent of the unconditional [Theorems 1.1](#) and [13.4](#). The next result keeps track of numerator-uniform constants. It requires an additional uniform saturation hypothesis because the proof above only absorbs Euler factors at the fixed prime set dividing m .

Hypothesis 13.5 (uniform numerator saturation package). *For every fixed $A > 0$, there is an exponent $B_A > 0$ such that the construction below is uniform for*

$$2 \leq m \leq (\log X)^A.$$

If [Hypothesis 13.8](#) is also assumed, we further assume that there are $\kappa_0, B > 0$ for which the same statement is uniform for $2 \leq m \leq X^{\kappa_0}$. In both ranges, use

$$d_- d_+ p \equiv -1 \pmod{m\rho(e)}, \quad b \equiv -me \pmod{d_-},$$

and omit variables sharing prime factors with m . We assume that the analogues of [Propositions 8.6](#) and [8.7](#) and [Lemmas 9.2](#) and [10.3](#) hold uniformly in the numerator and base class, with

$$\mu_{m,b} \asymp \frac{(\log X)^3}{\varphi(m)(m/\varphi(m))^B}.$$

Here $B = B_A$ in the logarithmic range, and the comparison constants and every $o(1)$ term are uniform over the stated range. The upper mass bound is part of the hypothesis because it enters the dependency and finite-level estimates. This uniform statement does not follow directly from the fixed-numerator theorem. It amounts to proving the saturation average, event tensorisation, appendage, and dependency bounds with all losses tracked uniformly in m . The Brun coefficient bound and finite-interval transfer then follow from these inputs exactly as in [Sections 9](#) and [11](#).

Theorem 13.6 (conditional growing-numerator Erdős–Straus–Schinzel exceptional sets). Assume [Hypothesis 13.5](#). For every fixed $A > 0$ there are constants $c_A, C_A, B_A > 0$ with the following property. For $N \geq 3$ and $2 \leq m \leq (\log N)^A$, put

$$\Xi_{A,m}(N) := \frac{(\log N)^{3/4}}{\varphi(m)(m/\varphi(m))^{B_A}}.$$

Then, uniformly in m ,

$$E_m(N) \leq C_A N \min \{1, (1 + \log \log(3N)) \exp\{-c_A \Xi_{A,m}(N)\}\}.$$

Proof. The divisor-fan certificate is unchanged except for replacing 4 by m . If

$$q = ma - 1, \quad e \mid a^2, \quad n \equiv -me \pmod{q},$$

and

$$x = an, \quad y = \frac{an + e}{q}, \quad z = \frac{an + a^2 n^2 / e}{q},$$

then, with $A_0 = an$,

$$\frac{1}{y} + \frac{1}{z} = \frac{q}{A_0}, \quad \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1+q}{an} = \frac{m}{n}.$$

Thus every such congruence is a valid certificate for m/n .

Run the saturated construction with the same exact divisors $e = rs^2$ and the same split $d = d_- d_+$, but replace the large-prime congruence and small-prime compatibility by

$$d_- d_+ p \equiv -1 \pmod{m\rho(e)}, \quad b \equiv -me \pmod{d_-}.$$

Then $q = d_- d_+ p$ has the form $q = ma - 1$ with $a = (q + 1)/m$, and the same congruence gives $\rho(e) \mid a$. The prime-distribution moduli become $m\rho(e)$. Because $m = X^{o(1)}$, these moduli are still $\leq X^{\theta+o(1)}$ and remain inside the Bombieri–Vinogradov range after decreasing the fixed parameter margin. After imposing $(m, \rho(e)) = 1$, the main prime-progression factor is

$$\frac{1}{\varphi(m\rho(e))} = \frac{1}{\varphi(m)\varphi(\rho(e))}.$$

[Hypothesis 13.5](#) supplies the required lower and upper mass scale, tensorisation, appendage, and dependency bounds with a loss bounded by a fixed power of $m/\varphi(m)$. The residual dependency graph still sees only the rough part d_+ and the large prime p , so large-prime uniqueness, the Brun coefficient estimate, Suen’s inequality, finite-interval transfer, and smooth lifting are identical to the proof of [Theorem 1.1](#) whenever the conditional mass tends to infinity. The recurrence for F_r , Suen’s bound, and the finite-interval transfer apply with μ_b replaced by $\mu_{m,b}$. In the smooth lift, the auxiliary scale M always satisfies $M > N^{1/2}$; hence $m \leq (\log N)^A \leq 2^A (\log M)^A$ remains in the logarithmic range covered by the hypothesis, and

$$\Xi_{A,m}(M) \geq 2^{-3/4} \Xi_{A,m}(N).$$

Also, at the chosen scale,

$$m \leq (\log N)^A \ll_A (\log X)^{4A} \leq (\log X)^{4A+1}$$

once X is large. Thus one fixed instance of [Hypothesis 13.5](#) is sufficient throughout. We apply that hypothesis with exponent $4A + 1$ and take the constant denoted by B_A in the theorem to be the resulting loss exponent B_{4A+1} . When the saving exponent is small, we retain the trivial estimate $E_m(N) \leq N$. We now choose

$$\log X = \alpha_A (\log N)^{1/4}$$

with $\alpha_A > 0$ small enough. The reduced no-hit estimate is $\ll_A N \exp\{-c_A \Xi_{A,m}(N)\}$. Summing over the smooth parts contributes

$$\prod_{p \leq z_N} \left(1 - \frac{1}{p}\right)^{-1} \ll 1 + \log \log(3N).$$

The range with rough part at most $N^{1/2}$ contributes $N^{3/4+o(1)}$, which is smaller than the asserted bound whenever the second term in the minimum is active. Combining this estimate with the trivial bound proves the displayed minimum uniformly in m . $\square$

For each fixed numerator $m \geq 4$, the unconditional [Theorem 13.4](#) already gives a saving exponent of order $(\log N)^{3/4}$, exceeding the $((\log N)^2/\varphi(m))^{1/3} = (\log N)^{2/3} \varphi(m)^{-1/3}$ of [13]. The following corollary extends this, *conditionally*, to a growing range of numerators.

Corollary 13.7 (conditional comparison with the Pomerance–Weingartner exponent). Assume [Hypothesis 13.5](#). Fix $\delta > 0$. For all sufficiently large N , uniformly for numerators m satisfying

$$4 \leq m \leq (\log N)^2, \quad \varphi(m) \leq (\log N)^{1/8-\delta},$$

the quantity $\Xi_{2,m}(N)$ in [Theorem 13.6](#) is larger than the Vaughan-type exponent

$$\left(\frac{(\log N)^2}{\varphi(m)} \right)^{1/3}$$

appearing in the Pomerance–Weingartner exceptional-set bound [13]. Consequently the conditional bound in [Theorem 13.6](#) gives a strictly stronger logarithmic saving throughout this range.

Proof. Quantitatively,

$$\frac{(\log N)^{3/4} / \{\varphi(m)(m/\varphi(m))^{B_2}\}}{((\log N)^2 / \varphi(m))^{1/3}} = \frac{(\log N)^{1/12}}{\varphi(m)^{2/3} (m/\varphi(m))^{B_2}} \geq (\log N)^{2\delta/3 - o(1)},$$

because $m/\varphi(m) \ll \log \log(3m)$. Thus the fan would give a strictly stronger logarithmic saving throughout this growing range of numerators whenever the stated uniform estimates are verified. In this range $\Xi_{2,m}(N)$ is a positive power of $\log N$, so the factor $1 + \log \log(3N)$ in [Theorem 13.6](#) is absorbed by decreasing the constant in the exponential. $\square$

Conditional distribution hypotheses

We use the following Elliott–Halberstam-strength reciprocal-prime distribution hypothesis; it is a weighted reciprocal-prime analogue of the usual level-of-distribution principle, not the standard Elliott–Halberstam statement itself [4].

Hypothesis 13.8 (reciprocal Elliott–Halberstam input). For every $\vartheta < 1$, every $A > 0$, and every fixed divisor-weight exponent $C \geq 0$, the reciprocal-prime Bombieri–Vinogradov estimate of [Lemma 7.9](#) holds with the moduli range $Q \leq T^\vartheta$ in place of the Bombieri–Vinogradov range $m \leq T^{1/2-\varepsilon}$. In the notation used here, we require that for every $A > 0$, every fixed divisor-weight exponent $C \geq 0$, and every $0 < \nu < \beta$, one has

$$\sum_{q \leq X^\nu} \tau(q)^C \max_{(a,q)=1} \left| \sum_{\substack{X^\beta < p \leq X^{1-\sigma} \\ p \equiv a \pmod{q}}} \frac{1}{p} - \frac{1}{\varphi(q)} \sum_{X^\beta < p \leq X^{1-\sigma}} \frac{1}{p} \right| \ll_{A,C,\nu} (\log X)^{-A}.$$

Thus this hypothesis replaces the restriction $\nu < \beta/2$ in [Lemma 7.9](#) by the full range $\nu < \beta$.

Theorem 13.9 (conditional numerator bound under Elliott–Halberstam and uniform saturation). Assume [Hypotheses 13.5](#) and [13.8](#). There are absolute constants $\kappa, c, C, B > 0$ with the following property. For $N \geq 3$ and

$$2 \leq m \leq \exp\{\kappa(\log N)^{1/4}\},$$

put

$$\Xi_m(N) := \frac{(\log N)^{3/4}}{\varphi(m)(m/\varphi(m))^B}.$$

Then, uniformly in m ,

$$E_m(N) \leq CN \min \{1, (1 + \log \log(3N)) \exp\{-c\Xi_m(N)\}\}.$$

In particular, for every fixed $\eta > 0$, this gives a nontrivial almost-all bound throughout the subfamily

$$m \leq \exp\{\kappa(\log N)^{1/4}\}, \quad \varphi(m)(m/\varphi(m))^B \leq (\log N)^{3/4-\eta},$$

where it gives $E_m(N) \ll N \exp\{-c_\eta(\log N)^\eta\}$.

Proof. We repeat the proof of [Theorem 13.6](#), with the prime-distribution margin supplied by [Hypothesis 13.8](#). Let κ_0 be the exponent in [Hypothesis 13.5](#). Choose $\kappa_1 > 0$ smaller than κ_0 and small enough that

$$\theta + \kappa_1 < \beta(1 - \varepsilon)$$

for some fixed $\varepsilon > 0$. If $m \leq X^{\kappa_1}$, then the prime progressions have modulus

$$m\rho(e) \leq X^{\theta+\kappa_1+o(1)} \leq (X^\beta)^{1-\varepsilon/2}.$$

Hypothesis 13.8 therefore supplies the required reciprocal-prime error bound, while **Hypothesis 13.5** supplies the mass, tensorisation, appendage, and dependency estimates uniformly in m .

Set $\log X = \alpha(\log N)^{1/4}$ as before. Choose the constant κ in the theorem so that

$$2^{1/4}\kappa/\alpha < \kappa_1.$$

Then $m \leq \exp\{\kappa(\log N)^{1/4}\}$ implies $m \leq X^{\kappa_1}$, and the same remains true at every auxiliary scale $M > N^{1/2}$ in the smooth lift, because $(\log N)^{1/4} \leq 2^{1/4}(\log M)^{1/4}$; also $\Xi_m(M) \geq 2^{-3/4}\Xi_m(N)$. The proof of **Theorem 13.6** now applies with the range $m \leq X^{\kappa_0}$ in the hypothesis. Its mass lower bound gives the displayed exponential term, and the smooth sum contributes the factor $1 + \log \log(3N)$. Combining this with the trivial bound gives the stated minimum. On the subfamily in the theorem, $\Xi_m(N) \geq (\log N)^\eta$, so the logarithmic factor is absorbed by decreasing c , which proves the stated almost-all consequence. $\square$

We close the conditional part with a template rather than a claimed construction. It records one sufficient fourth-coordinate packet: no duplication, a quartic mass law, controlled appendages, and small Suen dependencies. No such packet is constructed here.

Hypothesis 13.10 (speculative quartic Elliott–Halberstam saturated packet). *Put $z = (\log X)^5$. For every reduced base class $b \pmod{P(z)}$, assume that there is a finite family $\mathcal{I}_b^{(4)}(X)$ with the following data. Each i has a squarefree residual modulus $q_i \leq X$ satisfying $(q_i, P(z)) = 1$, a reduced residue $a_i \pmod{q_i}$, the event*

$$\mathcal{A}_i = \{n : n \equiv a_i \pmod{q_i}\}, \quad w_i = \frac{1}{q_i},$$

and a divisor-fan certificate valid for every integer in $\mathcal{A}_i \cap \{n : n \equiv b \pmod{P(z)}\}$. Two indices are called compatible when their residues agree modulo (q_i, q_j) . For compatible S , put

$$L_S = \text{lcm}(q_j : j \in S), \quad g(i; S) = (q_i, L_S).$$

On the product of the uniform residue spaces at all primes dividing the q_i , the events then have probability w_i , and a compatible intersection has probability $1/\text{lcm}(q_i : i \in S)$. We assume:

(i) **quartic mass:**

$$\mu_b^{(4)} := \sum_{i \in \mathcal{I}_b^{(4)}(X)} w_i \asymp (\log X)^4;$$

(ii) **EH-level prime distribution:** *for every $\vartheta < 1$, every $A > 0$, and every fixed divisor-weight exponent $C \geq 0$, the reciprocal-prime errors for the prime progressions occurring in the packet are summable over moduli up to the level T^ϑ with total error $O_{A,C,\vartheta}((\log T)^{-A})$;*

(iii) **uniform appendage:** *for every fixed $K > 0$ and every compatible S with $|S| \leq K\mu_b^{(4)}$,*

$$A(S) := \sum_{\substack{i \notin S \\ S \cup \{i\} \text{ compatible}}} w_i g(i; S) \leq \mu_b^{(4)}(1 + o_K(1));$$

(iv) **small dependencies:** *with $i \sim j$ meaning $i \neq j$ and $(q_i, q_j) > 1$,*

$$\delta_b^{(4)} = \sup_i \sum_{j \sim i} w_j = o(1), \quad \Delta_b^{(4)} = \sum_{\substack{\{i,j\}: i \sim j \\ i,j \text{ compatible}}} \frac{1}{\text{lcm}(q_i, q_j)} = o(\mu_b^{(4)}).$$

All asymptotic statements are uniform in the reduced base class b .

Theorem 13.11 (quartic optimization template under explicit packet hypotheses). *Assume the speculative packet in **Hypothesis 13.10**. Then there is a constant $c_4 > 0$ such that, for all $N \geq 3$,*

$$E(N) \ll N \exp\{-c_4(\log N)^{4/5}\}.$$

Proof. The appendage estimate gives, for $r \leq K\mu_b^{(4)}$,

$$F_r \leq \frac{(\mu_b^{(4)})^r}{r!} \exp\{o_K(\mu_b^{(4)})\}.$$

The dependency estimates and Suen's inequality give $\mathbb{P}(W_b = 0) \leq \exp\{-(1 - o(1))\mu_b^{(4)}\}$. Since the residual moduli are at most X and coprime to $P(z)$, the CRT row count and Bonferroni argument of **Section 11** apply without change.

Now choose $\log X = \alpha(\log N)^{1/5}$. Both terms in the logarithm of the finite level,

$$\log P(z) \ll z \asymp (\log X)^5, \quad \mu_b^{(4)} \log X \asymp (\log X)^5,$$

are at most a sufficiently small multiple of $\log N$ when α is small. Meanwhile $\mu_b^{(4)} \asymp (\log N)^{4/5}$. Summing over the reduced base classes and applying the smooth lift therefore gives

$$E(N) \ll N \exp\{-c_4(\log N)^{4/5}\}.$$

This proves the theorem. □

Remark 13.12. *Splitting an existing variable merely relabels the same exact divisor; the hypothesis requires a fourth independent conductor coordinate.*

14 Future directions

The cubic mass is intrinsic to the three-coordinate split $e = rs^2$, $d = d_-d_+$, $Q = dp$. The main open problem suggested by [Theorem 13.11](#) is to construct a *quartic* saturated packet unconditionally, which would push the exponent to $E(N) \ll N \exp\{-c_4(\log N)^{4/5}\}$; the obstruction is a fourth independent coordinate that retains logarithmic mass without inflating the dependency graph. A different route would replace the second-moment Brun–Suen input, which limits the per-class saving to $1 - o(1)$, by a higher-cluster correlation estimate controlling triple and higher overlaps directly. Separately, the growing-numerator results of [Section 13](#) are conditional on the uniform-saturation and reciprocal Elliott–Halberstam hypotheses; making either unconditional, even for a restricted range of m , would remove the conditionality from [Theorems 13.6](#) and [13.9](#).

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